Appendix 1: Second-Order Derivatives & Spherical Harmonics - Additional Material

It will first be demonstrated that a second-order pressure gradient microphone can not have a purely zeroth-order polar pattern. Such a polar pattern has Laplace series coefficients

$$A_0 = G \quad \text{(A1.1a)}$$

$$A_2 = 0 \quad \text{(A1.1b)}$$

$$A_{2,1} = 0 \quad \text{(A1.1c)}$$

$$A_{2,2} = 0 \quad \text{(A1.1d)}$$

$$B_{2,1} = 0 \quad \text{(A1.1e)}$$

$$B_{2,2} = 0 \quad \text{(A1.1f)}$$

It is not necessary to specify the values of the coefficients of the first-order spherical harmonics, since, as demonstrated in Chapter 3, these will necessarily be zero. Using the definitions established in Chapter 3, from equation (A1.1) we obtain

$$C_1 = G \quad \text{(A1.2a)}$$

$$C_2 = 0 \quad \text{(A1.2b)}$$

$$C_3 = 0 \quad \text{(A1.2c)}$$

$$C_4 = G \quad \text{(A1.2d)}$$

$$C_5 = 0 \quad \text{(A1.2e)}$$

$$C_6 = G \quad \text{(A1.2f)}$$

This requires that

$$x_1 x_2 = y_1 y_2 = z_1 z_2 = G \quad \text{(A1.3)}$$

and, since $C_2 = 0$,

$$x_1 y_2 + y_1 x_2 = 0 \quad \text{(A1.4)}$$

Suppose that $G > 0$. Then from equation (A1.3), x_1 and x_2 are both non-zero and have the same sign; similarly, both y_1 and y_2 are non-zero and have the same sign. But, from equation (A1.4)

$$\begin{aligned} x_1 y_2 &= -y_1 x_2 \\ \frac{x_1}{x_2} &= -\frac{y_1}{y_2} \end{aligned} \tag{A1.5}$$

The quotients exist, since x_2 and y_2 are both non-zero. As neither x_1 nor y_1 are zero, equation (A1.5) can only be satisfied if either x_1 and x_2 or y_1 and y_2 have opposite signs, but we have already established that this cannot be the case.

Assume instead that $G < 0$. Now x_1 and x_2 must have opposing signs, as must y_1 and y_2, but equation (A1.4), and hence equation (A1.5), must still hold; this is again a contradiction.

Therefore, a second-order pressure gradient microphone can not have a zeroth-order polar response pattern.

It will now be demonstrated that only the derivatives specified in Chapter 3 (or those trivially equivalent to them) can produce the second-order spherical harmonic polar patterns that are required.

We first consider the case

$$M(q, f) = \sin(2q)\cos^2(f) \tag{A1.6}$$

The Laplace series coefficients are

$$A_0 = 0 \tag{A1.7a}$$

$$A_2 = 0 \tag{A1.7b}$$

$$A_{2,1} = 0 \tag{A1.7c}$$

$$A_{2,2} = 0 \tag{A1.7d}$$

$$B_{2,1} = 0 \tag{A1.7e}$$

$$B_{2,2} = G \tag{A1.7f}$$

Hence

$$C_1 = 0 \tag{A1.8a}$$

$$C_2 = 6G \tag{A1.8b}$$

$$C_3 = 0 \tag{A1.8c}$$

$$C_4 = 0 \tag{A1.8d}$$

$$C_5 = 0 \tag{A1.8e}$$

$$C_6 = 0 \tag{A1.8f}$$

Since $C_1 = C_4 = C_6 = 0,$ so

$$x_1 x_2 = 0 \rightarrow x_1 = 0 \text{ or } x_2 = 0 \text{ or both} \tag{A1.9a}$$

$$y_1 y_2 = 0 \rightarrow y_1 = 0 \text{ or } y_2 = 0 \text{ or both} \tag{A1.9b}$$

$$z_1 z_2 = 0 \rightarrow z_1 = 0 \text{ or } z_2 = 0 \text{ or both} \tag{A1.9c}$$

and since $C_2 = 6G$, and $C_3 = C_5 = 0,$ so

$$x_1 y_2 + y_1 x_2 = 6G \tag{A1.10a}$$

$$x_1 z_2 + z_1 x_2 = 0 \tag{A1.10b}$$

$$y_1 z_2 + z_1 y_2 = 0 \tag{A1.10c}$$

Also, since $\hat{\mathbf{u}}_1$ and $\hat{\mathbf{u}}_2$ are unit vectors, so of course

$$x_1^{\,2} + y_1^{\,2} + z_1^{\,2} = 1 \tag{A1.11a}$$

$$x_2^{\,2} + y_2^{\,2} + z_2^{\,2} = 1 \tag{A1.11b}$$

From (A1.9), either x_1 or x_2 or both are equal to zero; similarly either y_1 or y_2 or both are equal to zero; but, since $G \neq 0$ (because we require the harmonic to be present), so from (A1.10a)

$$x_1 y_2 + y_1 x_2 \neq 0 \tag{A1.12}$$

Therefore, either $x_1 y_2 = 0$ and $y_1 x_2 \neq 0,$ or $x_1 y_2 \neq 0$ and $y_1 x_2 = 0.$ Hence, either

$$x_1 = 0, x_2 \neq 0, y_1 \neq 0, y_2 = 0$$

or

$$x_1 \neq 0, x_2 = 0, y_1 = 0, y_2 \neq 0$$

In either case, equations (A1.10b) and (A1.10c) must still be satisfied. Now, if $x_1 = y_2 = 0,$ then

$$x_1 z_2 + z_1 x_2 = z_1 x_2 = 0 \quad \text{(A1.13a)}$$
$$y_1 z_2 + z_1 y_2 = y_1 z_2 = 0 \quad \text{(A1.13b)}$$

and, since $y_1 \neq 0$ and $x_2 \neq 0,$ so $z_1 = z_2 = 0.$ If instead $x_2 = y_1 = 0,$ then

$$x_1 z_2 + z_1 x_2 = x_1 z_2 = 0 \quad \text{(A1.14a)}$$
$$y_1 z_2 + z_1 y_2 = z_1 y_2 = 0 \quad \text{(A1.14b)}$$

and, since $x_1 \neq 0$ and $y_2 \neq 0,$ again $z_1 = z_2 = 0.$ Thus, in either case,

$$z_1 = z_2 = 0 \quad \text{(A1.15)}$$

We now have

$$x_1 y_2 + y_1 x_2 = 6G \quad \text{(A1.16a)}$$
$$x_1^{\ 2} + y_1^{\ 2} = 1 \quad \text{(A1.16b)}$$
$$x_2^{\ 2} + y_2^{\ 2} = 1 \quad \text{(A1.16c)}$$

with either $x_1 = y_2 = 0$ or $x_2 = y_1 = 0.$ There exist a total of eight solutions:

$$\hat{\mathbf{u}}_1 = [1 \quad 0 \quad 0]^T \qquad \hat{\mathbf{u}}_2 = [0 \quad 1 \quad 0]^T \qquad G = \tfrac{1}{6} \quad \text{(A1.17a)}$$
$$\hat{\mathbf{u}}_1 = [1 \quad 0 \quad 0]^T \qquad \hat{\mathbf{u}}_2 = [0 \quad -1 \quad 0]^T \qquad G = -\tfrac{1}{6} \quad \text{(A1.17b)}$$

$$\hat{\mathbf{u}}_1 = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix}^T \qquad \hat{\mathbf{u}}_2 = \begin{bmatrix} 0 & -1 & 0 \end{bmatrix}^T \qquad G = \tfrac{1}{6} \tag{A1.17c}$$

$$\hat{\mathbf{u}}_1 = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix}^T \qquad \hat{\mathbf{u}}_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T \qquad G = -\tfrac{1}{6} \tag{A1.17d}$$

$$\hat{\mathbf{u}}_1 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T \qquad \hat{\mathbf{u}}_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T \qquad G = \tfrac{1}{6} \tag{A1.17e}$$

$$\hat{\mathbf{u}}_1 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T \qquad \hat{\mathbf{u}}_2 = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix}^T \qquad G = -\tfrac{1}{6} \tag{A1.17f}$$

$$\hat{\mathbf{u}}_1 = \begin{bmatrix} 0 & -1 & 0 \end{bmatrix}^T \qquad \hat{\mathbf{u}}_2 = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix}^T \qquad G = \tfrac{1}{6} \tag{A1.17g}$$

$$\hat{\mathbf{u}}_1 = \begin{bmatrix} 0 & -1 & 0 \end{bmatrix}^T \qquad \hat{\mathbf{u}}_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T \qquad G = -\tfrac{1}{6} \tag{A1.17h}$$

We now consider the case where

$$M(q, f) = \cos(2q)\cos^2(f) \tag{A1.18}$$

and the Laplace series coefficients are

$$A_0 = 0 \tag{A1.19a}$$

$$A_2 = 0 \tag{A1.19b}$$

$$A_{2,1} = 0 \tag{A1.19c}$$

$$A_{2,2} = G \tag{A1.19d}$$

$$B_{2,1} = 0 \tag{A1.19e}$$

$$B_{2,2} = 0 \tag{A1.19f}$$

so that

$$C_1 = 3G \tag{A1.20a}$$

$$C_2 = 0 \tag{A1.20b}$$

$$C_3 = 0 \tag{A1.20c}$$

$$C_4 = -3G \tag{A1.20d}$$

$$C_5 = 0 \tag{A1.20e}$$

$$C_6 = 0 \tag{A1.20f}$$

Now $G \neq 0$, because we require the harmonic to be present. From equations (A1.20a) and (A1.20d),

$$x_1 x_2 = 3G \tag{A1.21a}$$

$$y_1 y_2 = -3G \tag{A1.21b}$$

and none of x_1, x_2, y_1, y_2 can be zero.

Since $C_6 = 0$, so

$$z_1 z_2 = 0 \rightarrow z_1 = 0 \text{ or } z_2 = 0 \text{ or both} \tag{A1.22}$$

Further, since $C_2 = C_3 = C_5 = 0$, so

$$x_1 z_2 + z_1 x_2 = 0 \tag{A1.23a}$$

$$y_1 z_2 + z_1 y_2 = 0 \tag{A1.23b}$$

$$x_1 y_2 + y_1 x_2 = 0 \tag{A1.23c}$$

Consider equation (A1.23a). We know that either z_1 or z_2, or both, are equal to zero. If $z_1 = 0$ then $x_1 z_2 = 0$ and, since $x_1 \neq 0$, $z_2 = 0$. Similarly, if $z_2 = 0$ then $z_1 x_2 = 0$ and since $x_2 \neq 0$ so $z_1 = 0$. Hence

$$z_1 = z_2 = 0 \tag{A1.24}$$

and equations (A1.23a) and (A1.23b) reduce to $0 = 0$. Now since $\hat{\mathbf{u}}_1$ and $\hat{\mathbf{u}}_2$ are unit vectors,

$$x_1^2 + y_1^2 = 1 \tag{A1.25a}$$

$$x_2^2 + y_2^2 = 1 \tag{A1.25b}$$

which may be rearranged to give

$$x_1^2 = 1 - y_1^2 \tag{A1.26a}$$

$$x_2^2 = 1 - y_2^2 \tag{A1.26b}$$

or alternatively

$$y_1^{\;2} = 1 - x_1^{\;2} \tag{A1.27a}$$

$$y_2^{\;2} = 1 - x_2^{\;2} \tag{A1.27b}$$

Rearranging equation (A1.23c), we obtain

$$x_1 y_2 = -y_1 x_2 \tag{A1.28}$$

and squaring both sides gives

$$x_1^{\;2} y_2^{\;2} = y_1^{\;2} x_2^{\;2} \tag{A1.29}$$

Substituting in from equations (A1.27a) and (A1.27b) gives

$$\begin{aligned} x_1^{\;2}\left(1 - x_2^{\;2}\right) &= \left(1 - x_1^{\;2}\right) x_2^{\;2} \\ x_1^{\;2} - x_1^{\;2} x_2^{\;2} &= x_2^{\;2} - x_1^{\;2} x_2^{\;2} \\ x_1^{\;2} &= x_2^{\;2} \end{aligned} \tag{A1.30}$$

from which it follows that

$$|x_1| = |x_2| \tag{A1.31}$$

Similarly, we can instead obtain

$$\begin{aligned} \left(1 - y_1^{\;2}\right) y_2^{\;2} &= y_1^{\;2}\left(1 - y_2^{\;2}\right) \\ y_2^{\;2} - y_1^{\;2} y_2^{\;2} &= y_1^{\;2} - y_1^{\;2} y_2^{\;2} \\ y_2^{\;2} &= y_1^{\;2} \end{aligned} \tag{A1.32}$$

from which it follows that

$$|y_1| = |y_2| \tag{A1.33}$$

Now from equations (A1.21a) and (A1.31)

$$\begin{aligned} |x_1 x_2| &= |x_1||x_2| \\ &= |x_1|^2 \\ &= 3|G| \end{aligned} \tag{A1.34}$$

and so

$$|x_1|^2 = 3|G| \tag{A1.35}$$

Similarly,

$$|y_1|^2 = 3|G| \tag{A1.36}$$

Therefore

$$|x_1| = |x_2| = |y_1| = |y_2| \tag{A1.37}$$

Now from equation (A1.25a),

$$|x_1|^2 + |y_1|^2 = 1 \tag{A1.38}$$

and, since $|x_1| = |y_1|$, so

$$\begin{aligned} 2|x_1|^2 &= 1 \\ |x_1|^2 &= \frac{1}{2} \\ |x_1| &= \frac{1}{\sqrt{2}} \end{aligned} \tag{A1.39}$$

and therefore, from equation (A1.37),

$$|x_1| = |x_2| = |y_1| = |y_2| = \frac{1}{\sqrt{2}} \tag{A1.40}$$

and

$$\begin{aligned}|G| &= \tfrac{1}{3}|x_1|^2 \\ &= \tfrac{1}{6}\end{aligned} \tag{A1.41}$$

so that

$$G = \pm\tfrac{1}{6} \tag{A1.42}$$

Now, if $G = \frac{1}{6}$, then x_1 and x_2 have the same sign, while y_1 and y_2 have opposite signs; if $G = -\frac{1}{6}$, then x_1 and x_2 have opposite signs, while y_1 and y_2 have the same sign. We obtain again a total of eight solutions:

$$\hat{\mathbf{u}}_1 = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\end{bmatrix}^T \qquad \hat{\mathbf{u}}_2 = \begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\end{bmatrix}^T \qquad G = \tfrac{1}{6} \tag{A1.43a}$$

$$\hat{\mathbf{u}}_1 = \begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\end{bmatrix}^T \qquad \hat{\mathbf{u}}_2 = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\end{bmatrix}^T \qquad G = \tfrac{1}{6} \tag{A1.43b}$$

$$\hat{\mathbf{u}}_1 = \begin{bmatrix}-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\end{bmatrix}^T \qquad \hat{\mathbf{u}}_2 = \begin{bmatrix}-\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\end{bmatrix}^T \qquad G = \tfrac{1}{6} \tag{A1.43c}$$

$$\hat{\mathbf{u}}_1 = \begin{bmatrix}-\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\end{bmatrix}^T \qquad \hat{\mathbf{u}}_2 = \begin{bmatrix}-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\end{bmatrix}^T \qquad G = \tfrac{1}{6} \tag{A1.43d}$$

$$\hat{\mathbf{u}}_1 = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\end{bmatrix}^T \qquad \hat{\mathbf{u}}_2 = \begin{bmatrix}-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\end{bmatrix}^T \qquad G = -\tfrac{1}{6} \tag{A1.43e}$$

$$\hat{\mathbf{u}}_1 = \begin{bmatrix}-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\end{bmatrix}^T \qquad \hat{\mathbf{u}}_2 = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\end{bmatrix}^T \qquad G = -\tfrac{1}{6} \tag{A1.43f}$$

$$\hat{\mathbf{u}}_1 = \begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\end{bmatrix}^T \qquad \hat{\mathbf{u}}_2 = \begin{bmatrix}-\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\end{bmatrix}^T \qquad G = -\tfrac{1}{6} \tag{A1.43g}$$

$$\hat{\mathbf{u}}_1 = \begin{bmatrix}-\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\end{bmatrix}^T \qquad \hat{\mathbf{u}}_2 = \begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\end{bmatrix}^T \qquad G = -\tfrac{1}{6} \tag{A1.43h}$$

For the cases $M(q,f)=\cos(q)\sin(2f)$ and $M(q,f)=\sin(q)\sin(2f)$, the method is exactly parallel to that described for $M(q,f)=\sin(2q)\cos^2(f)$.

For $M(q,f)=\cos(q)\sin(2f)$, we obtain the results

$$\hat{\mathbf{u}}_1=\begin{bmatrix}0 & 0 & 1\end{bmatrix}^T \quad \hat{\mathbf{u}}_2=\begin{bmatrix}1 & 0 & 0\end{bmatrix}^T \quad G=\tfrac{1}{3} \tag{A1.44a}$$
$$\hat{\mathbf{u}}_1=\begin{bmatrix}0 & 0 & 1\end{bmatrix}^T \quad \hat{\mathbf{u}}_2=\begin{bmatrix}-1 & 0 & 0\end{bmatrix}^T \quad G=-\tfrac{1}{3} \tag{A1.44b}$$
$$\hat{\mathbf{u}}_1=\begin{bmatrix}0 & 0 & -1\end{bmatrix}^T \quad \hat{\mathbf{u}}_2=\begin{bmatrix}1 & 0 & 0\end{bmatrix}^T \quad G=-\tfrac{1}{3} \tag{A1.44c}$$
$$\hat{\mathbf{u}}_1=\begin{bmatrix}0 & 0 & -1\end{bmatrix}^T \quad \hat{\mathbf{u}}_2=\begin{bmatrix}-1 & 0 & 0\end{bmatrix}^T \quad G=\tfrac{1}{3} \tag{A1.44d}$$
$$\hat{\mathbf{u}}_1=\begin{bmatrix}1 & 0 & 0\end{bmatrix}^T \quad \hat{\mathbf{u}}_2=\begin{bmatrix}0 & 0 & 1\end{bmatrix}^T \quad G=\tfrac{1}{3} \tag{A1.44e}$$
$$\hat{\mathbf{u}}_1=\begin{bmatrix}1 & 0 & 0\end{bmatrix}^T \quad \hat{\mathbf{u}}_2=\begin{bmatrix}0 & 0 & -1\end{bmatrix}^T \quad G=-\tfrac{1}{3} \tag{A1.44f}$$
$$\hat{\mathbf{u}}_1=\begin{bmatrix}-1 & 0 & 0\end{bmatrix}^T \quad \hat{\mathbf{u}}_2=\begin{bmatrix}0 & 0 & 1\end{bmatrix}^T \quad G=-\tfrac{1}{3} \tag{A1.44g}$$
$$\hat{\mathbf{u}}_1=\begin{bmatrix}-1 & 0 & 0\end{bmatrix}^T \quad \hat{\mathbf{u}}_2=\begin{bmatrix}0 & 0 & -1\end{bmatrix}^T \quad G=\tfrac{1}{3} \tag{A1.44h}$$

while for $M(q,f)=\sin(q)\sin(2f)$, the solutions are

$$\hat{\mathbf{u}}_1=\begin{bmatrix}0 & 1 & 0\end{bmatrix}^T \quad \hat{\mathbf{u}}_2=\begin{bmatrix}0 & 0 & -1\end{bmatrix}^T \quad G=-\tfrac{1}{3} \tag{A1.45a}$$
$$\hat{\mathbf{u}}_1=\begin{bmatrix}0 & -1 & 0\end{bmatrix}^T \quad \hat{\mathbf{u}}_2=\begin{bmatrix}0 & 0 & 1\end{bmatrix}^T \quad G=-\tfrac{1}{3} \tag{A1.45b}$$
$$\hat{\mathbf{u}}_1=\begin{bmatrix}0 & 1 & 0\end{bmatrix}^T \quad \hat{\mathbf{u}}_2=\begin{bmatrix}0 & 0 & 1\end{bmatrix}^T \quad G=\tfrac{1}{3} \tag{A1.45c}$$
$$\hat{\mathbf{u}}_1=\begin{bmatrix}0 & -1 & 0\end{bmatrix}^T \quad \hat{\mathbf{u}}_2=\begin{bmatrix}0 & 0 & -1\end{bmatrix}^T \quad G=\tfrac{1}{3} \tag{A1.45d}$$
$$\hat{\mathbf{u}}_1=\begin{bmatrix}0 & 0 & 1\end{bmatrix}^T \quad \hat{\mathbf{u}}_2=\begin{bmatrix}0 & -1 & 0\end{bmatrix}^T \quad G=-\tfrac{1}{3} \tag{A1.45e}$$
$$\hat{\mathbf{u}}_1=\begin{bmatrix}0 & 0 & -1\end{bmatrix}^T \quad \hat{\mathbf{u}}_2=\begin{bmatrix}0 & 1 & 0\end{bmatrix}^T \quad G=-\tfrac{1}{3} \tag{A1.45f}$$
$$\hat{\mathbf{u}}_1=\begin{bmatrix}0 & 0 & 1\end{bmatrix}^T \quad \hat{\mathbf{u}}_2=\begin{bmatrix}0 & 1 & 0\end{bmatrix}^T \quad G=\tfrac{1}{3} \tag{A1.45g}$$
$$\hat{\mathbf{u}}_1=\begin{bmatrix}0 & 0 & -1\end{bmatrix}^T \quad \hat{\mathbf{u}}_2=\begin{bmatrix}0 & -1 & 0\end{bmatrix}^T \quad G=\tfrac{1}{3} \tag{A1.45h}$$

The final second-order spherical harmonic polar pattern is

$$M(q,f)=\frac{1}{2}\left(3\sin^2(f)-1\right) \tag{A1.46}$$

for which the Laplace series coefficients are

$$A_0 = 0 \tag{A1.47a}$$

$$A_2 = G \tag{A1.47b}$$

$$A_{2,1} = 0 \tag{A1.47c}$$

$$A_{2,2} = 0 \tag{A1.47d}$$

$$B_{2,1} = 0 \tag{A1.47e}$$

$$B_{2,2} = 0 \tag{A1.47f}$$

so that

$$C_1 = -\tfrac{1}{2}G \tag{A1.48a}$$

$$C_2 = 0 \tag{A1.48b}$$

$$C_3 = 0 \tag{A1.48c}$$

$$C_4 = -\tfrac{1}{2}G \tag{A1.48d}$$

$$C_5 = 0 \tag{A1.48e}$$

$$C_6 = G \tag{A1.48f}$$

Now $G \neq 0,$ because we require the harmonic to be present; hence, none of $x_1,$ $x_2,$ $y_1,$ $y_2,$ $z_1,$ z_2 can be equal to zero.

Since $C_2 = C_3 = C_5 = 0,$ so

$$x_1 y_2 + y_1 x_2 = 0 \tag{A1.49a}$$

$$x_1 z_2 + z_1 x_2 = 0 \tag{A1.49b}$$

$$y_1 z_2 + z_1 y_2 = 0 \tag{A1.49c}$$

and, rearranging equation (A1.49a),

$$x_1 y_2 = -y_1 x_2 \tag{A1.50}$$

Since $x_1 \neq 0$ and $y_1 \neq 0,$ from equations (A1.48a) and (A1.48d) we may write

$$x_2 = -\frac{G}{2x_1} \tag{A1.51a}$$

$$y_2 = -\frac{G}{2y_1} \tag{A1.51b}$$

Substituting these in equation (A1.50) gives

$$\begin{aligned} x_1\left(-\frac{G}{2y_1}\right) &= -y_1\left(-\frac{G}{2x_1}\right) \\ -\frac{G}{2}\frac{x_1}{y_1} &= \frac{G}{2}\frac{y_1}{x_1} \\ -\frac{x_1}{y_1} &= \frac{y_1}{x_1} \\ -x_1^{\ 2} &= y_1^{\ 2} \end{aligned} \tag{A1.52}$$

which has no real solutions except for $x_1 = y_1 = 0,$ which has already been eliminated as a possible solution.

The desired polar response cannot therefore be obtained as a derivative in this way. Instead, we seek a derivative which will generate the required polar response plus an omnidirectional component (which can be removed if the pressure is available as a separate signal). We thus set

$$A_0 = \overline{G} \tag{A1.53a}$$

$$A_2 = G \tag{A1.53b}$$

$$A_{2,1} = 0 \tag{A1.53c}$$

$$A_{2,2} = 0 \tag{A1.53d}$$

$$B_{2,1} = 0 \tag{A1.53e}$$

$$B_{2,2} = 0 \tag{A1.53f}$$

so that

$$C_1 = \overline{G} - \tfrac{1}{2}G \tag{A1.54a}$$

$$C_2 = 0 \tag{A1.54b}$$

$$C_3 = 0 \tag{A1.54c}$$

$$C_4 = \overline{G} - \tfrac{1}{2}G \tag{A1.54d}$$

$$C_5 = 0 \tag{A1.54e}$$

$$C_6 = \overline{G} + G \tag{A1.54f}$$

Since $C_1 = C_4 = \overline{G} - \frac{1}{2}G$ and $C_6 = \overline{G} + G$, so

$$x_1 x_2 = \overline{G} - \tfrac{1}{2}G \tag{A1.55a}$$

$$y_1 y_2 = \overline{G} - \tfrac{1}{2}G \tag{A1.55b}$$

$$z_1 z_2 = \overline{G} + G \tag{A1.55c}$$

Neither G nor $\overline{G}$ is equal to zero, since both harmonics must be present.

Since $C_2 = C_3 = C_5 = 0$,

$$x_1 y_2 + y_1 x_2 = 0 \tag{A1.56a}$$

$$x_1 z_2 + z_1 x_2 = 0 \tag{A1.56b}$$

$$y_1 z_2 + z_1 y_2 = 0 \tag{A1.56c}$$

From equations (A1.55a) and (A1.55b),

$$x_1 x_2 = y_1 y_2 \tag{A1.57}$$

and from equation (A1.56a)

$$x_1 y_2 = -y_1 x_2 \tag{A1.58}$$

Hence

$$\left.\begin{aligned} x_1 &= \frac{y_1 y_2}{x_2} \\ x_1 &= -\frac{y_1 x_2}{y_2} \end{aligned}\right\} \to \frac{y_1 y_2}{x_2} = -\frac{y_1 x_2}{y_2} \to \frac{y_2}{x_2} = -\frac{x_2}{y_2} \tag{A1.59a}$$

$$\left.\begin{aligned} x_2 &= \frac{y_1 y_2}{x_1} \\ x_2 &= -\frac{x_1 y_2}{y_1} \end{aligned}\right\} \to \frac{y_1 y_2}{x_1} = -\frac{x_1 y_2}{y_1} \to \frac{y_1}{x_1} = -\frac{x_1}{y_1} \tag{A1.59b}$$

$$\left.\begin{aligned} y_1 &= \frac{x_1 x_2}{y_2} \\ y_1 &= -\frac{x_1 y_2}{x_2} \end{aligned}\right\} \to \frac{x_1 x_2}{y_2} = -\frac{x_1 y_2}{x_2} \to \frac{x_2}{y_2} = -\frac{y_2}{x_2} \tag{A1.59c}$$

$$\left.\begin{aligned} y_2 &= \frac{x_1 x_2}{y_1} \\ y_2 &= -\frac{y_1 x_2}{x_1} \end{aligned}\right\} \to \frac{x_1 x_2}{y_1} = -\frac{y_1 x_2}{x_1} \to \frac{x_1}{y_1} = -\frac{y_1}{x_1} \tag{A1.59d}$$

Since x_1, x_2, y_1 and y_2 must all be real, these results all lead to contradictions; for any real number *x*, *x* and $1/x$ have the same sign. We must therefore make at least one element zero; the quotients will then not exist and the contradiction will not arise. Since at least one of x_1, x_2, y_1, and y_2 must be zero, so at least one of the products $x_1 x_2$ and $y_1 y_2$ must be zero; from equation (A1.57) we therefore have

$$x_1 x_2 = 0 \to x_1 = 0 \text{ or } x_2 = 0 \text{ or both} \tag{A1.60a}$$

$$y_1 y_2 = 0 \to y_1 = 0 \text{ or } y_2 = 0 \text{ or both} \tag{A1.60b}$$

Furthermore, from equations (A1.55a) and (A1.60a),

$$\begin{aligned}\bar{G} - \tfrac{1}{2}G &= 0 \\ G &= 2\bar{G}\end{aligned} \quad \text{(A1.61)}$$

Substituting this into equation (A1.55c) gives

$$z_1 z_2 = 3\bar{G} \quad \text{(A1.62)}$$

Since neither G nor $\bar{G}$ is zero, neither z_1 nor z_2 can be equal to zero. Now, since neither z_1 nor z_2 can be zero, and at least one of x_1 and x_2 and at least one of y_1 and y_2 must be equal to zero, equations (A1.56b) and (A1.56c) can only be satisfied if

$$x_1 = x_2 = y_1 = y_2 = 0 \quad \text{(A1.63)}$$

This leaves

$$z_1^{\ 2} = 1 \rightarrow z_1 = \pm 1 \quad \text{(A1.64a)}$$

$$z_2^{\ 2} = 1 \rightarrow z_2 = \pm 1 \quad \text{(A1.64b)}$$

and

$$\begin{aligned}3\bar{G} &= z_1 z_2 \\ &= \pm 1 \\ \bar{G} &= \pm \tfrac{1}{3}\end{aligned} \quad \text{(A1.65)}$$

for which there exist four solutions, rather than eight as in the previous cases:

$$\hat{\mathbf{u}}_1 = \begin{bmatrix}0 & 0 & 1\end{bmatrix}^T \quad \hat{\mathbf{u}}_2 = \begin{bmatrix}0 & 0 & 1\end{bmatrix}^T \quad G = \tfrac{2}{3} \quad \bar{G} = \tfrac{1}{3} \quad \text{(A1.66a)}$$

$$\hat{\mathbf{u}}_1 = \begin{bmatrix}0 & 0 & 1\end{bmatrix}^T \quad \hat{\mathbf{u}}_2 = \begin{bmatrix}0 & 0 & -1\end{bmatrix}^T \quad G = -\tfrac{2}{3} \quad \bar{G} = -\tfrac{1}{3} \quad \text{(A1.66b)}$$

$$\hat{\mathbf{u}}_1 = \begin{bmatrix}0 & 0 & -1\end{bmatrix}^T \quad \hat{\mathbf{u}}_2 = \begin{bmatrix}0 & 0 & 1\end{bmatrix}^T \quad G = -\tfrac{2}{3} \quad \bar{G} = -\tfrac{1}{3} \quad \text{(A1.66c)}$$

$$\hat{\mathbf{u}}_1 = \begin{bmatrix}0 & 0 & -1\end{bmatrix}^T \quad \hat{\mathbf{u}}_2 = \begin{bmatrix}0 & 0 & -1\end{bmatrix}^T \quad G = \tfrac{2}{3} \quad \bar{G} = \tfrac{1}{3} \quad \text{(A1.66d)}$$

Appendix 2: Ambisonic Signal Formats

A number of different signal sets are or may be used at one stage or another of an ambisonic system. For convenient reference, these various signal sets are briefly described in this appendix.

A2.1: A-, B-, C- & D-Format

As discussed in the main body of this thesis, the output signals of the microphone capsules making up a soundfield microphone are referred to as A-format signals. This signal set is not available to the "outside world"; it is utilised only within the soundfield microphone itself.

The B-format signal set, which is the primary signal format for ambisonic use, is also described in detail in the main text.

Ideally, the B-format signals would be communicated directly to the listener. Unfortunately, this has not always been possible - in particular, the need to distribute recordings via two-channel media, retaining compatibility with existing stereo and mono equipment, led to the need for alternative signal formats to be employed. The signal set which is conveyed to the listener via a recording or transmission medium, when it differs from B-format, is termed C-format. The "C" is sometimes said to stand for "consumer"; the term "consumer-format" does make sense in this context, although one suspects that this nomenclature post-dates the C-format designation. The C-format signal sets which were proposed as part of the initial development of ambisonics are together known as the UHJ hierarchy and are described in the next section.

The loudspeaker feed signals produced by an ambisonic decoder are sometimes referred to as D-format signals [8].

A2.2: The UHJ Hierarchy

The UHJ set, or hierarchy, of signal formats was designed to allow the use of two-channel recording and transmission media, while providing mono and stereo compatibility and allowing an upgrade to "full" ambisonic reproduction where circumstances and facilities permitted [21] [22] [48].

The UHJ hierarchy consists of four signal formats: BHJ, SHJ, THJ and PHJ. Since only BHJ encoded material was ever released commercially, the terms UHJ and BHJ have tended to be used synonymously, and in fact UHJ is generally thought to mean BHJ.

BHJ consists of two signals, derived from the *W*, *X* and *Y* signals of B-format and optimised for mono and two-channel stereo compatibility. The two signals are denoted *S* and *D*; *S* is the monophonic signal, while for two-speaker stereo reproduction left and right speaker feeds *L* and *R* are obtained as

$$L = \tfrac{1}{2}(S + D) \tag{A2.1a}$$

$$R = \tfrac{1}{2}(S - D) \tag{A2.1b}$$

A suitable decoder can extract approximations to the original signals *W*, *X* and *Y* and thus produce speaker feed signals for pantophonic surround sound reproduction, although there is of course some information loss due to the matrixing of three signals into two.

A third signal *T* may be added to the BHJ set. A suitable decoder can then use these three signals to exactly reconstruct the original pantophonic B-format signals; however, the stereo and mono compatibility of the BHJ format is left unchanged. If *T* is of full bandwidth, then this signal set is known as THJ; a reduced bandwidth *T* signal may also be employed, in which case we obtain the SHJ signal set. Hence SHJ provides some improvement in directional effect compared to BHJ, while THJ provides full B-format and thus a further improvement. Since neither SHJ nor THJ has ever been utilised commercially, it is unlikely that any confusion will result from the use of *T* to designate both this signal and part of the second-order B-format set.

A fourth signal, *Q*, may be added to the THJ set, giving PHJ. *Q* is obtained directly from the B-format *Z* signal and thus carries height information; hence, PHJ provides for full first-order periphonic ambisonic reproduction. Alternatively, the *Q* signal may be used to create a loudspeaker emphasis effect - something generally contrary to the goals of ambisonic reproduction, but nonetheless desired by some content producers.

A detailed description of the UHJ encoding specification may be found in [48].

A2.3: Enhanced B-Format; BE-, BF- & BEF-Format

Enhanced B-format signal sets have been proposed in connection with B-format decoders optimised specifically for use with HDTV, or more generally for use in support of visual media. The primary motivation is to produce a frontal sound stage which is more stable with respect to movement by the listener, and specifically to “lock” centre-front acoustic images in place with respect to a screen. Thus, the motivation is substantially the same as for the use of the centre channel in cinema-oriented surround sound formats.

Two extra signals are defined, denoted ***E*** and ***F***. These signals have directional response patterns:

$$M_E(q) = \begin{cases} \frac{K_e}{\sqrt{2}}\left(1 - K_g(1-\cos(q))\right) & |q| \le q_S \\ 0 & \text{otherwise} \end{cases} \tag{A2.2a}$$

$$M_F(q) = \begin{cases} K_f \sin(q) & |q| \le q_S \\ -K_b \sin(q) & |180^\circ - q| \le q_B \\ 0 & \text{otherwise} \end{cases} \tag{A2.2b}$$

where q_S is the half-stage angular width for the frontal sound stage, typically in the range 60° to 70°, q_B is similarly the half-stage width for the rear sound stage, K_g has a value of 3.25 ± 0.25, K_e is a parameter with a value between 0 and 1 controlling the impact of the ***E*** signal, and similarly K_f and K_b control the effect of the ***F*** signal.

Adding the ***E*** signal to the B-format set produces BE-format; similarly inclusion of ***F*** results in BF-format, while including both ***E*** and ***F*** gives the BEF-format signal set. These signals sets are described in [48] [52].

A2.4: G-Format & “G+2”

As mentioned in Chapter 1, it has recently been suggested that ambisonic recordings could be distributed in a pre-decoded form, targeted at the standard 5-channel surround sound speaker layout [22]. These pre-decoded distribution signals are referred to as G-format signals. G-format has the significant advantage that the listener does not need to be equipped with an ambisonic decoder, although other advantages of ambisonics, such as the inclusion of height information and the ability to decode for the local loudspeaker layout, are lost. The conversion from B-format to G-format is however reversible, so that the suitably equipped listener may recover the B-format signals and use them to drive an ambisonic decoder. It has been suggested that as well as carrying G-format, a DVD might also carry BHJ encoded ambisonic material as a stereo soundtrack; the G-format and BHJ signals may be obtained from the same B-format mix [22]. This combination has been called “G+2” format.