3: Theory of Pressure Gradient Microphones

In an isotropic medium such as air, the sound pressure at a point due to the propagation of a sound wave is independent of the direction in which the wave travels through that point [11] [89]; hence, a transducer which responds to the sound pressure at a point is omnidirectional, responding equally to waves incident from any direction. Real pressure microphones only approximate this ideal, since they are not truly "point" devices. Such a microphone responds to the pressure over its diaphragm, and an omnidirectional characteristic is obtained only because sound waves diffract around the capsule so that the pressure on any face of the capsule does not depend on the direction of incidence. At frequencies where the dimensions of the capsule are comparable to the wavelength, acoustic shadowing occurs and such microphones tend to become directional, responding more to frontally incident waves and less to sound arriving from other directions. Nevertheless, it is possible to construct small pressure microphones that are substantially omnidirectional over the full audio bandwidth; that is, from 20 Hz to 20 kHz.

The earliest microphones were nominally omnidirectional, pressure-responding devices. For many recording applications, it is desirable that microphones should not be omnidirectional, but should have other directional characteristics. The first directional microphones appear to have been the ribbon microphones developed at RCA [69]; the first of these was made available as a commercial product in 1931 [73]. Such microphones are useful because they allow the sound from a specific source to be more readily distinguished from ambient or "background" sound, which often constitutes unwanted interference [72] [74] [76] [81]. They are also required to facilitate the coincident microphone techniques often used when making stereo and surround sound recordings [13] [17] [42] [53] [75].

Directional microphones may be classified as being of either the "interference" / "wave" or the "pressure gradient" type [70] [74] [76] [78]. Pressure gradient devices are by far the most commonly used, and will be discussed here in detail. They respond to the differences in sound pressure between two or more points separated by distances which are small compared to the wavelength.

By contrast, interference microphones must have dimensions at least comparable to, and usually larger than, the wavelength. This class includes such devices as parabolic reflectors, acoustic lenses, large-surface microphones, and "line" (also known as "shotgun" and "interference tube") microphones; of these, only the line microphone is in common use [76]. Because of the long wavelengths associated with the lower reaches of the audio frequency range (approximately 1.7 m at 200 Hz), it is not usually practical to employ such microphones when pickup over the full audio bandwidth is required [72] [74] [76]. Furthermore, the polar patterns exhibit sidelobes which vary with frequency, so that off-axis comb filtering effects occur; these are usually sufficient to render microphones of this type unsuitable for high quality recording [37] [72].

3.1: First-Order Pressure Gradient Microphones

The theory of first-order pressure gradient microphones will now be developed. Proofs that specific types of microphones (e.g., the ribbon microphone) respond to the pressure gradient are abundant in the literature [11] [62] [69] [71], and such standard material will not be duplicated here. Rather, the following discussion is concerned with results which follow from the concept of measuring the pressure gradient, regardless of the transduction mechanism employed.

By "pressure gradient" is meant the rate of change of sound pressure with respect to a (small) movement in space; that is, the derivative of sound pressure with respect to displacement. That displacement may be in an arbitrary direction, which will depend on the orientation of the microphone. In the opinion of the author, the directional derivative is therefore the obvious mathematical tool to utilise for the analysis of such microphones. However, this method has not generally been adopted in the literature (although one occasionally finds that substantially this approach has been employed, but without explicit use of the notation of vector calculus; e.g., [25]). Although the results presented in this section are not themselves new, it may be claimed that the derivations are more satisfactory and more general than those available in the existing literature. In addition, the treatment here of first-order microphones will provide the basis for a similar treatment of second-order pressure gradient microphones which, except for certain special cases, have received little attention to date.

Let p be the sound pressure at a point $\mathbf{x}$ in space defined in terms of some "world" coordinate system. Consider a plane wave propagating in a direction specified by spherical polar coordinate angles $(\bar{q}, \bar{f})$, as shown in figure 3.1.

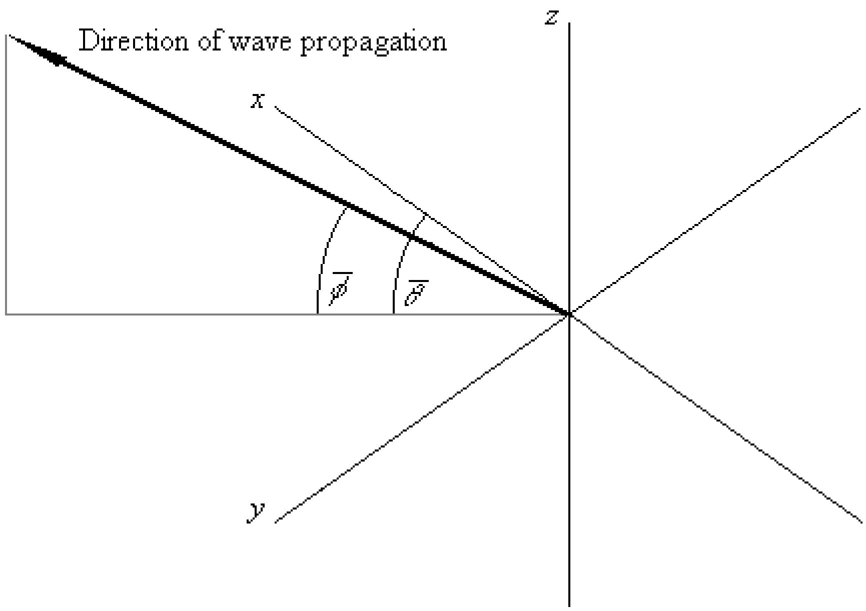


Figure 3.1: Direction of Wave Propagation in "World" Coordinate System

Let **k** be the wave vector

$$\mathbf{k} = k\begin{bmatrix} \cos(\bar{q})\cos(\bar{f}) \\ \sin(\bar{q})\cos(\bar{f}) \\ \sin(\bar{f}) \end{bmatrix} \tag{3.1}$$

where the wave number

$$k = \frac{2p}{l} = \frac{w}{c} \tag{3.2}$$

Using the complex phasor notation

$$\begin{aligned} p &= Ae^{j(wt - \mathbf{k}\cdot\mathbf{x})} \\ &= Ae^{j(wt - k\cos(\bar{q})\cos(\bar{f})x - k\sin(\bar{q})\cos(\bar{f})y - k\sin(\bar{f})z)} \end{aligned} \tag{3.3}$$

and

$$
\begin{aligned}
\nabla p &= \begin{bmatrix} \dfrac{\partial p}{\partial x} \\ \dfrac{\partial p}{\partial y} \\ \dfrac{\partial p}{\partial z} \end{bmatrix} \\
&= -jkA \begin{bmatrix} \cos(\bar{q})\cos(\bar{f}) \\ \sin(\bar{q})\cos(\bar{f}) \\ \sin(\bar{f}) \end{bmatrix} e^{j(wt - k\cos(\bar{q})\cos(\bar{f})x - k\sin(\bar{q})\cos(\bar{f})y - k\sin(\bar{f})z)} \\
&= -jk \begin{bmatrix} \cos(\bar{q})\cos(\bar{f}) \\ \sin(\bar{q})\cos(\bar{f}) \\ \sin(\bar{f}) \end{bmatrix} p
\end{aligned}
\tag{3.4}
$$

Now let $\hat{\mathbf{u}}_1$ be a unit vector pointing in a direction defined by spherical polar coordinate angles (q', f');

$$
\hat{\mathbf{u}}_1 = \begin{bmatrix} \cos(q')\cos(f') \\ \sin(q')\cos(f') \\ \sin(f') \end{bmatrix}
\tag{3.5}
$$

The directional derivative of p in the direction $\hat{\mathbf{u}}_1$ is then

$$
\begin{aligned}
\hat{\mathbf{u}}_1 \cdot \nabla p &= -jk \begin{bmatrix} \cos(\bar{q})\cos(\bar{f}) \\ \sin(\bar{q})\cos(\bar{f}) \\ \sin(\bar{f}) \end{bmatrix} \cdot \begin{bmatrix} \cos(q')\cos(f') \\ \sin(q')\cos(f') \\ \sin(f') \end{bmatrix} p \\
&= -jk[\cos(\bar{q})\cos(\bar{f})\cos(q')\cos(f') + \sin(\bar{q})\cos(\bar{f})\sin(q')\cos(f') \\
&\qquad + \sin(\bar{f})\sin(f')]p \\
&= -jk[\cos(\bar{f})\cos(f')\cos(\bar{q} - q') + \sin(\bar{f})\sin(f')]p
\end{aligned}
\tag{3.6}
$$

The factor $[\cos(\bar{f})\cos(f')\cos(\bar{q} - q') + \sin(\bar{f})\sin(f')]$ is the scalar product of the two unit vectors $\hat{\mathbf{u}}_1$ and $(1/k)\mathbf{k}$, and is therefore equal to the cosine of the angle between them. Thus, it is equal to the cosine of the angle between the direction in which $\hat{\mathbf{u}}_1$ points and the direction of propagation of the wave.

This result has been expressed in terms of a “world” coordinate system, in which a microphone may be placed at an arbitrary position, but it is conventional and often more useful to work in terms of a coordinate system centred on the microphone. It is convenient in this case to describe the plane wave in terms of the direction from which it is incident, rather than the direction in which it propagates. Let the wave be incident from a direction (q, f), as shown in figure 3.2.

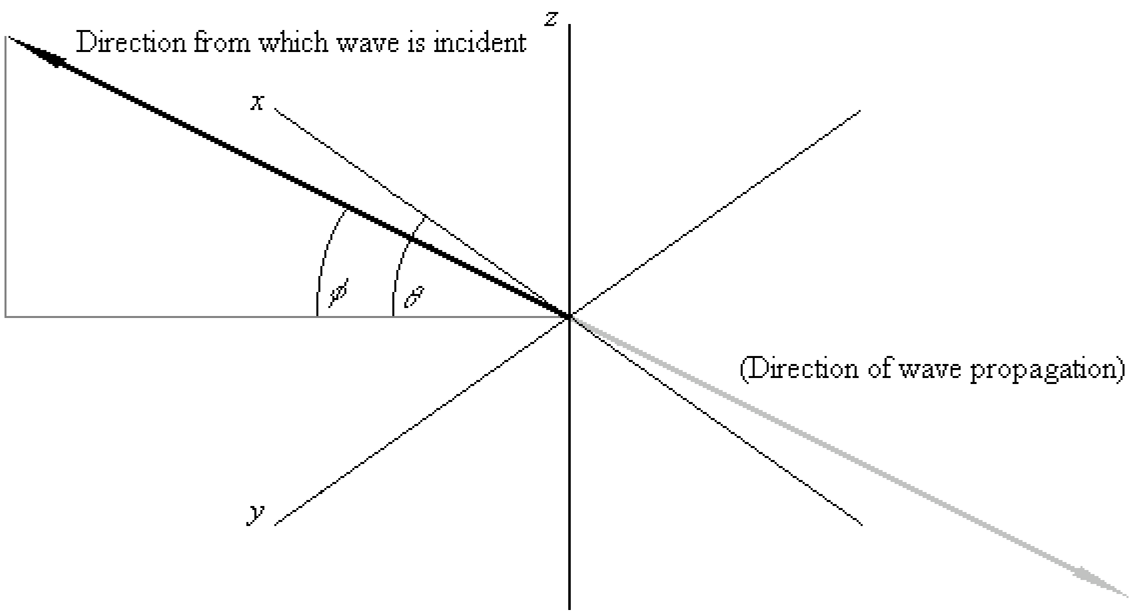


Figure 3.2: Direction of Incidence in Microphone Coordinate System

We define the “wave incidence vector”

$$\tilde{\mathbf{k}} = k \begin{bmatrix} \cos(q)\cos(f) \\ \sin(q)\cos(f) \\ \sin(f) \end{bmatrix} \tag{3.7}$$

and the sound pressure at the point $\mathbf{x}_m$ is then given by

$$p = Ae^{j(wt + \tilde{\mathbf{k}} \cdot \mathbf{x}_m)} \tag{3.8}$$

(note the change of sign in the exponent), so that

$$\nabla p = jk \begin{bmatrix} \cos(q)\cos(f) \\ \sin(q)\cos(f) \\ \sin(f) \end{bmatrix} p \tag{3.9}$$

Let $\hat{\mathbf{u}}_1$ point in the direction given in the microphone coordinate system by angles (q', f'). Then the directional derivative

$$\hat{\mathbf{u}}_1 \cdot \nabla p = jk[\cos(f)\cos(f')\cos(q - q') + \sin(f)\sin(f')]p \tag{3.10}$$

It is convenient to orient the microphone coordinate system such that $\hat{\mathbf{u}}_1$ is coincident with the *x* axis, i.e., so that $\hat{\mathbf{u}}_1 = \hat{\mathbf{x}}$. (This is consistent with the convention that the *x* axis points forwards, since $\hat{\mathbf{u}}_1$ defines the direction in which the microphone points.) In this case, $q' = f' = 0$ and

$$\hat{\mathbf{u}}_1 \cdot \nabla p = jk\cos(q)\cos(f)p \tag{3.11}$$

The amplitude of this pressure gradient depends on the direction from which the wave is incident, but is also proportional to frequency; additionally, the pressure gradient leads the sound pressure by 90°. These (unsurprising) consequences of differentiation can be eliminated by appropriate filtering; specifically, by integration with respect to time and multiplication by *c*;

$$\begin{aligned} c\int(\hat{\mathbf{u}}_1 \cdot \nabla p)dt &= c\frac{1}{jw} jk\cos(q)\cos(f)p \\ &= \frac{ck}{w}\cos(q)\cos(f)p \\ &= \cos(q)\cos(f)p \end{aligned} \tag{3.12}$$

Note that the dependence of the amplitude on frequency, and the phase shift, are eliminated by the integration; multiplication by *c* simply removes a constant scale factor that would otherwise appear. It is included here for convenience, but in a real microphone capsule it may be considered to be one factor contributing to the total responsivity; hence, it is not of fundamental importance, and the fact that the velocity of sound may vary slightly with atmospheric conditions is not significant so far as the present exposition of theory is concerned. This will be referred to henceforth as the "equalised (pressure) gradient response". In most cases, the integration is inherent in the operation of the transducer. The polar pattern of this microphone is

$$M(q, f) = \cos(q)\cos(f) \tag{3.13}$$

This is the well-known dipole, "figure-of-eight", or "bidirectional" polar response associated with a pressure gradient microphone. It has one positive and one negative lobe; positive in this context implies that the output is in phase with (of the same polarity as) the sound pressure, while negative indicates that the output is in antiphase with (of opposite polarity to) the pressure. The two-dimensional (planar) polar pattern is obtained by setting $f = 0$ so that $\cos(f) = 1$ and

$$M(q) = \cos(q) \tag{3.14}$$

To obtain other directional responses, a microphone which responds to a weighted sum of pressure and equalised pressure gradient may be used; proofs that specific capsule designs have such a response may again be found in the literature [11] [57] [71] [77] [93]. The output of such a microphone is given by

$$v = G\frac{1}{a+b}\left[ap + bc\int(\hat{\mathbf{u}}_1 \cdot \nabla p)dt\right] \tag{3.15}$$

or, by substituting from equation (3.12),

$$\begin{aligned} v &= G\frac{1}{a+b}\left[a + b\cos(q)\cos(f)\right]p \\ &= GM(q, f)\,p \end{aligned} \tag{3.16}$$

where a and b are constants that define the ratio of the pressure and equalised pressure gradient components, G is an overall responsivity constant, and $M(q,f)$ is the polar pattern

$$M(q,f) = \frac{1}{a+b}\left[a + b\cos(q)\cos(f)\right] \tag{3.17}$$

M is normalised to unity in the direction of maximum response; this direction, defined by $\hat{\mathbf{u}}_1$, is often termed the “directivity axis” of the microphone. The polar patterns of all first-order microphones exhibit “axial symmetry”; the three-dimensional pattern may be obtained by rotating the two-dimensional response about the directivity axis.

Equation (3.16) may equivalently be written as

$$v = G\frac{1}{a+b}\left[a + b\hat{\mathbf{u}}_1 \cdot \hat{\mathbf{d}}\right]p \tag{3.18}$$

where $\hat{\mathbf{d}}$ is a unit vector pointing in the direction from which the wave is incident, so that

$$\tilde{\mathbf{k}} = k\hat{\mathbf{d}} \tag{3.19}$$

and

$$M(q,f) = \frac{1}{a+b}\left[a + b\hat{\mathbf{u}}_1 \cdot \hat{\mathbf{d}}\right] \tag{3.20}$$

This form is convenient when working with several microphone capsules within a common coordinate system, and for that reason will be useful later in this thesis.

Henceforth, a “pure” first-order gradient microphone, as described by equation (3.12), will be described as a “first-order gradient microphone”, while a microphone of the type described by equation (3.15) will be termed a “first-order microphone”. Hence, a first-order gradient microphone is a first-order microphone with $a = 0$.

When $a = b = 1$, a so-called “cardioid” (implying “heart-shaped”) response is obtained

$$v_{cardioid} = \frac{G}{2}[1 + \cos(q)\cos(f)]p \tag{3.21}$$

while putting $a = 1$ and $b = 3$ results in a “hypercardioid” pattern

$$v_{hypercardioid} = \frac{G}{4}[1 + 3\cos(q)\cos(f)]p \tag{3.22}$$

A “supercardioid” is obtained by setting $a = 3$ and $b = 5,$ while any pattern for which $a > b$ is termed “subcardioid”. Plots of cardioid, hypercardioid, supercardioid and representative subcardioid polar responses are given in figure 3.3.

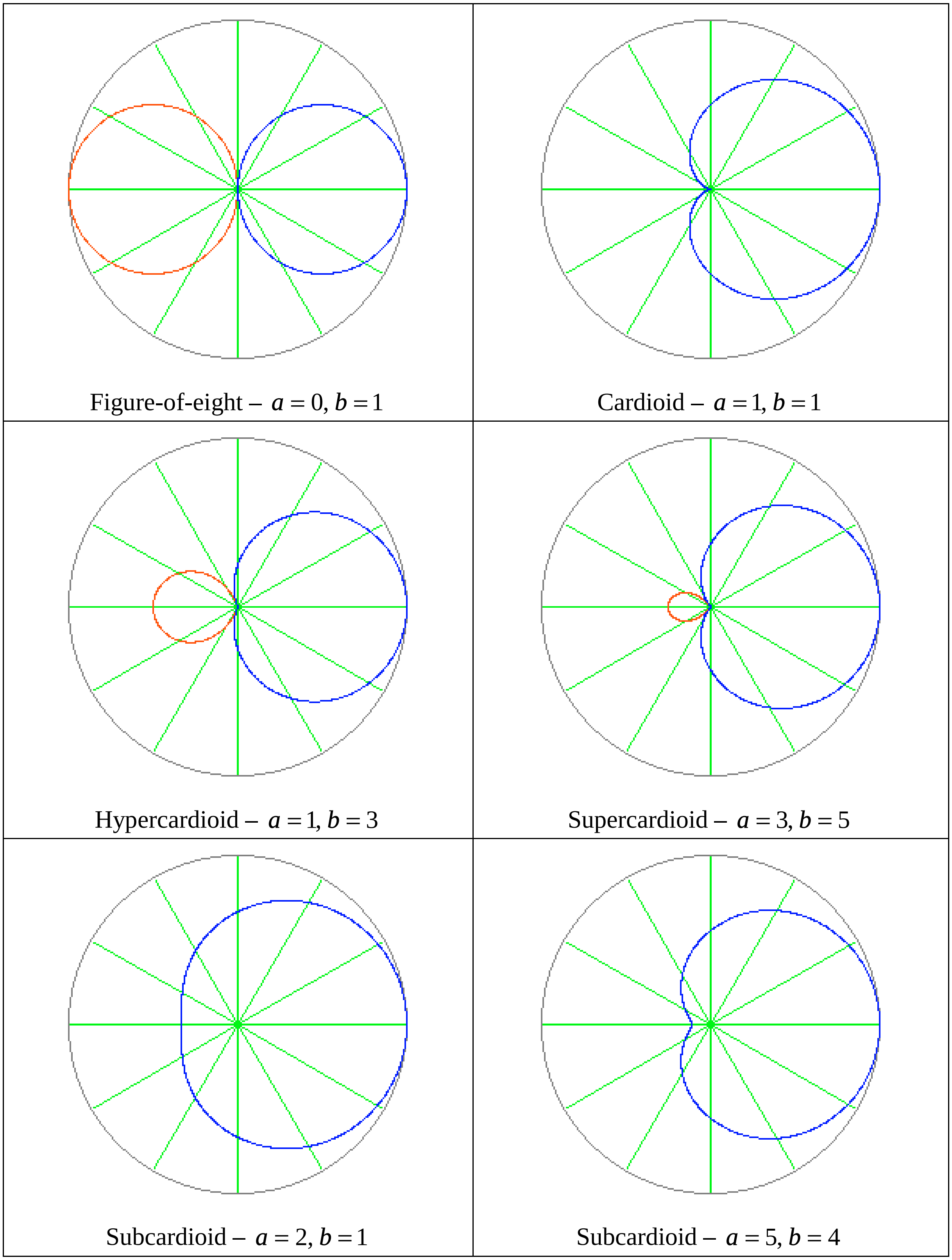
Use of this nomenclature is not entirely standardised; some authors describe as hypercardioid or supercardioid any microphone where $a < b,$ distinguishing between different patterns by the angles at which the nulls occur or by the ratio of the front and back responses; hence, a microphone with $a = 1$ and $b = 3$ may be described as a “110°-null hypercardioid” or as a “6 dB front-to-back hypercardioid”. Cardioid microphones are sometimes described as “unidirectional”, although this is rather difficult to justify, since a cardioid response is only 6 dB down at 90° off axis.

In the case of a real microphone capsule, the values of G, a, and b may be frequency-dependent, so that the responsivity and / or the polar pattern vary as functions of frequency; such behaviour is usually considered to be undesirable [37] [74] [93].

We observe that the polar pattern of an omnidirectional microphone is a zeroth-order spherical harmonic, while from equation (3.13) it can be seen that the polar response of a first-order pressure gradient microphone is a first-order spherical harmonic. Furthermore, from equation (3.17) it can be seen that the polar response of a first-order microphone responding to a weighted sum of pressure and pressure gradient is the equivalently weighted sum of a zeroth-order and a first-order spherical harmonic.

If a first-order figure-of-eight microphone is rotated so that $\hat{\mathbf{u}}_1 = \hat{\mathbf{y}},$ then the resulting polar pattern is

$$M_Y(q, f) = \sin(q)\cos(f) \tag{3.23}$$



Blue → positive (in phase)

Red → negative (antiphase)

Figure 3.3: Representative First-Order Polar Patterns

while if $\hat{\mathbf{u}}_1 = \hat{\mathbf{z}}$ then the polar response is

$$M_Z(q, f) = \sin(f) \tag{3.24}$$

Now, any other value of $\hat{\mathbf{u}}_1$ may be expressed as a linear combination of $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$. It follows that the polar response of any first-order gradient microphone may be expressed as a linear combination of $\cos(q)\cos(f)$, $\sin(q)\cos(f)$ and $\sin(f)$; that is, as a linear combination of the three independent first-order spherical harmonics. Furthermore, if the outputs of three co-located mutually orthogonal first-order gradient microphones are available, then the output of a notional gradient microphone in any orientation may be synthesised by forming a suitable linear combination of the three capsule output signals. Since the response of a general first-order microphone, as in equation (3.15), consists of the weighted sum of an omnidirectional component and a first-order gradient component, it further follows that the polar response of any first-order microphone may be described using the zeroth-order and three first-order spherical harmonics, and that, given the outputs of an omnidirectional microphone and three mutually orthogonal pure first-order gradient microphones, all co-located, the output of a notional first-order microphone of arbitrary polar pattern and orientation may be synthesised. It will be appreciated that it is not in reality possible to arrange for such a set of microphones to be exactly co-located; this will be discussed in a later chapter.

3.2: Second-Order Pressure Gradient Microphones

The second-order directional derivative of sound pressure is given by $\hat{\mathbf{u}}_2 \cdot \nabla(\hat{\mathbf{u}}_1 \cdot \nabla p)$; it is the directional derivative in the direction $\hat{\mathbf{u}}_2$ of the scalar field which is the directional derivative in the direction $\hat{\mathbf{u}}_1$ of the sound pressure. In general, $\hat{\mathbf{u}}_1 \neq \hat{\mathbf{u}}_2$; however, we will consider first the case where the two vectors are equal. Let $\hat{\mathbf{u}}_1 = \hat{\mathbf{u}}_2 = \hat{\mathbf{x}}$ in the microphone coordinate system, then

$$\begin{aligned}\hat{\mathbf{u}}_2 \cdot \nabla(\hat{\mathbf{u}}_1 \cdot \nabla p) &= \hat{\mathbf{x}} \cdot \nabla(\hat{\mathbf{x}} \cdot \nabla p) \\ &= \hat{\mathbf{x}} \cdot \nabla(jk \cos(q)\cos(f)\, p) \\ &= jk \cos(q)\cos(f)\hat{\mathbf{x}} \cdot \nabla p \\ &= (jk)^2 \cos^2(q)\cos^2(f)\, p \\ &= -k^2 \cos^2(q)\cos^2(f)\, p\end{aligned} \tag{3.25}$$

As one would expect, equalisation of a second-order gradient response requires a filter which is a double integrator. The microphone output is therefore given by

$$v = c^2 \int\int \left(\hat{\mathbf{u}}_2 \cdot \nabla(\hat{\mathbf{u}}_1 \cdot \nabla p)\right) dtdt \tag{3.26}$$

Applying such equalisation, we obtain, in the current case, an output signal

$$v = \cos^2(q)\cos^2(f)\, p \tag{3.27}$$

Microphones having such axial quadrupole, or second-order figure-of-eight, polar responses were first described by Olson [70] [76]. Note that the pattern has two lobes, but as well as being narrower than in the first-order case, they are both positive.

If two second-order figure-of-eight microphones are positioned with their directivity axes oriented respectively in the *x* and *y* directions and their outputs are added, the resulting signal is

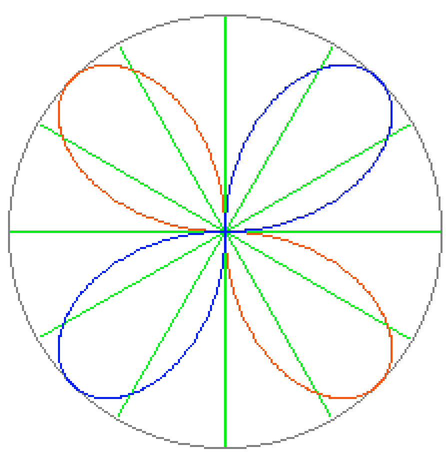
$$\begin{aligned}v &= c^2 \int\int \left(\hat{\mathbf{x}} \cdot \nabla(\hat{\mathbf{x}} \cdot \nabla p) + \hat{\mathbf{y}} \cdot \nabla(\hat{\mathbf{y}} \cdot \nabla p)\right) dtdt \\ &= \left[\cos^2(q)\cos^2(f) + \sin^2(q)\cos^2(f)\right] p \\ &= \cos^2(f)\, p\end{aligned} \tag{3.28}$$

Microphones having this second-order toroidal polar pattern have been investigated in the context of their potential application to conference telephony [25] [78] [80] [82]. It is possible to construct first-order toroidal microphones, but these require broadband 90° phase-shift circuits and have a phase response that varies with the direction of incidence [95].

We now consider a case where $\hat{\mathbf{u}}_1 \neq \hat{\mathbf{u}}_2$. Let $\hat{\mathbf{u}}_1 = \hat{\mathbf{x}}$ and $\hat{\mathbf{u}}_2 = \hat{\mathbf{y}}$; then the microphone output

$$
\begin{aligned}
v &= c^2 \iint \left(\hat{\mathbf{y}} \cdot \nabla (\hat{\mathbf{x}} \cdot \nabla p) \right) dtdt \\
&= c^2 \iint \left(\hat{\mathbf{y}} \cdot \nabla (jk \cos(q) \cos(f) p) \right) dtdt \\
&= c^2 \iint \left(jk \cos(q) \cos(f) \hat{\mathbf{y}} \cdot \nabla p \right) dtdt \\
&= c^2 \iint \left(jk \cos(q) \cos(f) \times jk \sin(q) \cos(f) \right) p\,dtdt \qquad (3.29) \\
&= c^2 \iint \left(-k^2 \right) \cos(q) \sin(q) \cos^2(f)\, p\,dtdt \\
&= \cos(q) \sin(q) \cos^2(f)\, p \\
&= \frac{1}{2} \sin(2q) \cos^2(f)\, p
\end{aligned}
$$

This exhibits the tesseral quadrupole, or “clover-leaf”, polar pattern considered by Gerzon [37] [38] [39]. The planar tesseral quadrupole pattern is shown in figure 3.4; note that axial symmetry is not present, and that the three-dimensional response can not therefore be obtained by rotation of the planar polar pattern. It may not be immediately apparent that this directional response is very useful and, indeed, in isolation it probably is not. Note, however, that it corresponds to one of the second-order spherical harmonics; it will be demonstrated later that pickup patterns of this form are useful in conjunction with other microphones having different polar patterns, and are also specifically required for second-order ambisonic recording.



Blue → positive (in phase)

Red → negative (antiphase)

Figure 3.4: Planar Tesseral Quadrupole (“Clover-Leaf”) Polar Pattern

A very large number of polar patterns may be obtained by combining zeroth-, first-, and second-order components. Extending equation (3.15) to include a second-order gradient term as in equation (3.26), we obtain for the output of such a microphone

$$v = G\frac{1}{k}\left[ap + b_1 c\int\left(\hat{\mathbf{u}}_{1,1}\cdot\nabla p\right)dt + b_2 c^2\int\int\left(\hat{\mathbf{u}}_{2,2}\cdot\nabla(\hat{\mathbf{u}}_{2,1}\cdot\nabla p)\right)dtdt\right] \tag{3.30}$$

where the constants a, b_1 and b_2, which specify the relative weightings of the pressure, first-order gradient and second-order gradient components, as well as the three direction vectors, may all be chosen independently. The normalising constant $1/k$ has the same purpose as the $1/(a+b)$ factor in the first-order case, but it is not necessarily equal to $1/(a+b_1+b_2)$.

If all three direction vectors are equal, $a=0$, and $b_1=b_2$, then the polar pattern obtained is the sum of a co-located first-order and second-order figure-of-eight with a common axis and equal axial responsivity;

$$\begin{aligned} v &= G\frac{1}{2}\left[\cos(q)\cos(f)+\cos^2(q)\cos^2(f)\right]p \\ &= \frac{G}{2}\left[1+\cos(q)\cos(f)\right]\cos(q)\cos(f)\,p \end{aligned} \tag{3.31}$$

This polar pattern is sometimes referred to as “second-order unidirectional”, although this term has also been applied to polar responses achieved by combining the first-order and second-order gradient elements in different ratios, possibly including a pressure component as well [70] [74] [79] [81]. The term “second-order cardioid” is also used, although the “heart” shape is no longer apparent. It will be noted that this response is very much more directional than the first-order cardioid, having nulls at 90° and 180°, and very small (negative) rear lobes (see figure 3.6 at the end of this section).

It has previously been shown that the output of any first-order microphone can be synthesised by taking a linear combination of the outputs of four coincident microphones having appropriate polar patterns. We now consider the extension of this concept to second-order microphones.

Let

$$\hat{\mathbf{u}}_1 = [x_1 \quad y_1 \quad z_1]^T \tag{3.32}$$

and

$$\hat{\mathbf{u}}_2 = [x_2 \quad y_2 \quad z_2]^T \tag{3.33}$$

We may then write

$$\begin{aligned}
\hat{\mathbf{u}}_2 \cdot \nabla(\hat{\mathbf{u}}_1 \cdot \nabla p) &= \hat{\mathbf{u}}_2 \cdot \nabla\left(x_1 \frac{\partial p}{\partial x} + y_1 \frac{\partial p}{\partial y} + z_1 \frac{\partial p}{\partial z} \right) \\
&= x_2 \frac{\partial}{\partial x}\left\{ x_1 \frac{\partial p}{\partial x} + y_1 \frac{\partial p}{\partial y} + z_1 \frac{\partial p}{\partial z} \right\} + y_2 \frac{\partial}{\partial y}\left\{ x_1 \frac{\partial p}{\partial x} + y_1 \frac{\partial p}{\partial y} + z_1 \frac{\partial p}{\partial z} \right\} \\
&\quad + z_2 \frac{\partial}{\partial z}\left\{ x_1 \frac{\partial p}{\partial x} + y_1 \frac{\partial p}{\partial y} + z_1 \frac{\partial p}{\partial z} \right\} \\
&= x_1 x_2 \frac{\partial^2 p}{\partial x^2} + (x_1 y_2 + y_1 x_2)\frac{\partial^2 p}{\partial x \partial y} + (x_1 z_2 + z_1 x_2)\frac{\partial^2 p}{\partial x \partial z} + y_1 y_2 \frac{\partial^2 p}{\partial y^2} \\
&\quad + (y_1 z_2 + z_1 y_2)\frac{\partial^2 p}{\partial y \partial z} + z_1 z_2 \frac{\partial^2 p}{\partial z^2}
\end{aligned} \tag{3.34}$$

Any second-order pressure gradient may thus be expressed as a linear combination of six second-order partial derivatives. In the plane wave case

$$\frac{\partial^2 p}{\partial x^2} = -k^2 \cos^2(q)\cos^2(f)\, p \tag{3.35a}$$

$$\begin{aligned}
\frac{\partial^2 p}{\partial x \partial y} &= -k^2 \cos(q)\sin(q)\cos^2(f)\, p \\
&= -\tfrac{1}{2} k^2 \sin(2q)\cos^2(f)\, p
\end{aligned} \tag{3.35b}$$

$$\begin{aligned}
\frac{\partial^2 p}{\partial x \partial z} &= -k^2 \cos(q)\cos(f)\sin(f)\, p \\
&= -\tfrac{1}{2} k^2 \cos(q)\sin(2f)\, p
\end{aligned} \tag{3.35c}$$

$$\frac{\partial^2 p}{\partial y^2} = -k^2 \sin^2(q)\cos^2(f)\, p \tag{3.35d}$$

$$\begin{aligned}
\frac{\partial^2 p}{\partial y \partial z} &= -k^2 \sin(q)\cos(f)\sin(f)\, p \\
&= -\tfrac{1}{2} k^2 \sin(q)\sin(2f)\, p
\end{aligned} \tag{3.35e}$$

$$\frac{\partial^2 p}{\partial^2 z} = -k^2 \sin^2(f)\, p \qquad (3.35f)$$

and so

$$\begin{aligned}\hat{\mathbf{u}}_2 \cdot \nabla(\hat{\mathbf{u}}_1 \cdot \nabla p) = -k^2 \big[& x_1 x_2 \cos^2(q)\cos^2(f) + (x_1 y_2 + y_1 x_2)\cos(q)\sin(q)\cos^2(f) \\ & + (x_1 z_2 + z_1 x_2)\cos(q)\cos(f)\sin(f) + y_1 y_2 \sin^2(q)\cos^2(f) \\ & + (y_1 z_2 + z_1 y_2)\sin(q)\cos(f)\sin(f) + z_1 z_2 \sin^2(f) \big] p \\ = -k^2 M(q,f)\, p & \end{aligned} \qquad (3.36)$$

where

$$\begin{aligned} M(q,f) = & C_1 \cos^2(q)\cos^2(f) + C_2 \cos(q)\sin(q)\cos^2(f) \\ & + C_3 \cos(q)\cos(f)\sin(f) + C_4 \sin^2(q)\cos^2(f) \\ & + C_5 \sin(q)\cos(f)\sin(f) + C_6 \sin^2(f) \end{aligned} \qquad (3.37)$$

with

$$C_1 = x_1 x_2 \qquad (3.38a)$$

$$C_2 = x_1 y_2 + y_1 x_2 \qquad (3.38b)$$

$$C_3 = x_1 z_2 + z_1 x_2 \qquad (3.38c)$$

$$C_4 = y_1 y_2 \qquad (3.38d)$$

$$C_5 = y_1 z_2 + z_1 y_2 \qquad (3.38e)$$

$$C_6 = z_1 z_2 \qquad (3.38f)$$

The Laplace series expansion of $M(q,f)$ has coefficients

$$A_0 = \tfrac{1}{3}C_1 + \tfrac{1}{3}C_4 + \tfrac{1}{3}C_6 \qquad (3.39a)$$

$$A_1 = 0 \qquad (3.39b)$$

$$A_2 = \tfrac{2}{3}C_6 - \tfrac{1}{3}C_1 - \tfrac{1}{3}C_4 \qquad (3.39c)$$

$$A_{1,1} = 0 \qquad (3.39d)$$

$$A_{2,1} = \tfrac{1}{3}C_3 \qquad (3.39e)$$

$$A_{2,2} = \tfrac{1}{6}C_1 - \tfrac{1}{6}C_4 \tag{3.39f}$$

$$B_{1,1} = 0 \tag{3.39g}$$

$$B_{2,1} = \tfrac{1}{3}C_5 \tag{3.39h}$$

$$B_{2,2} = \tfrac{1}{6}C_2 \tag{3.39i}$$

(This may be established either by using the formulæ given in Chapter 2 (equation (2.10)), or by manipulation using trigonometric identities.) Note that the three first-order coefficients are equal to zero, regardless of the values of the elements of $\hat{\mathbf{u}}_1$ and $\hat{\mathbf{u}}_2$; hence, a second-order gradient microphone never has a first-order spherical harmonic component in its polar response. Note also that

$$\begin{aligned} A_0 &= \tfrac{1}{3}(x_1x_2 + y_1y_2 + z_1z_2) \\ &= \tfrac{1}{3}\hat{\mathbf{u}}_1 \cdot \hat{\mathbf{u}}_2 \end{aligned} \tag{3.40}$$

This shows that the polar pattern of a pure second-order gradient microphone can include a zeroth-order spherical harmonic and, indeed, will always exhibit such a component unless $\hat{\mathbf{u}}_1 \cdot \hat{\mathbf{u}}_2 = 0$; i.e., unless the two vectors are perpendicular. It is not possible, however, for a single second-order gradient microphone to have a purely zeroth-order polar pattern (this is proved in Appendix 1).

There are only five second-order spherical harmonics in the Laplace series. Hence, the six partial derivatives which appear in equation (3.34) result in an expression with only five second-order terms, along with a zeroth-order term. This can be accounted for by the trigonometric identity

$$\cos^2(q)\cos^2(f) + \sin^2(q)\cos^2(f) + \sin^2(f) \equiv 1 \tag{3.41}$$

or by noting that a plane wave satisfies the Helmholtz equation

$$\nabla^2 p = -k^2 p \tag{3.42}$$

where

$$\nabla^2 p \equiv \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} \tag{3.43}$$

It should be noted that if the sound field does not satisfy equation (3.42) then equation (3.34) cannot be expressed in terms of zeroth-order and second-order spherical harmonics in the manner described above, since the sound pressure and second-order derivatives can not then be assumed to be linearly dependent; this is an important observation, which has significant implications for second-order ambisonic systems, and will be discussed in more detail in subsequent chapters. It should also be noted that no further linear dependencies can exist between the sound pressure and the second-order partial derivatives, because the spherical harmonics are linearly independent.

The polar patterns associated with the partial derivatives in equations (3.35b), (3.35c) and (3.35e) are second-order spherical harmonics. It may be seen that the remaining two second-order spherical harmonics are associated with combinations of pressure and second derivatives such that

$$\frac{\partial^2 p}{\partial x^2} - \frac{\partial^2 p}{\partial y^2} = -k^2 \cos(2q)\cos^2(f)\, p \tag{3.44a}$$

$$\frac{1}{2}\left(3\frac{\partial^2 p}{\partial z^2} + k^2 p\right) = -\frac{1}{2}k^2\left(3\sin^2(f) - 1\right)p \tag{3.44b}$$

In Appendix 1, it is proved that no other second-order partial derivatives of sound pressure result in polar responses which are second-order spherical harmonics.

Given an omnidirectional pressure microphone and five second-order pressure gradient microphones with polar responses corresponding to the five second-order spherical harmonics, all co-located, any possible second-order gradient microphone in any orientation may by synthesised by forming a suitable linear combination of output signals. If three first-order pressure gradient microphones corresponding to the first-order spherical harmonics are also included, then any possible second-order microphone of the form represented by equation (3.30) may be synthesised. Note, however, that while the spherical harmonic decomposition of a polar response is unique, there is often more than one way in which that polar pattern may be obtained in terms of the second-order partial derivatives; that is, there may exist a multiplicity of second-order pressure gradient microphone arrangements which give the same response to a plane wave, but cannot be assumed to give equal outputs in response to arbitrary sound fields. For example, the toroidal response of equation (3.28) may also be obtained using an omnidirectional microphone and an axial quadrupole with $\hat{\mathbf{u}}_1 = \hat{\mathbf{u}}_2 = \hat{\mathbf{z}}$.

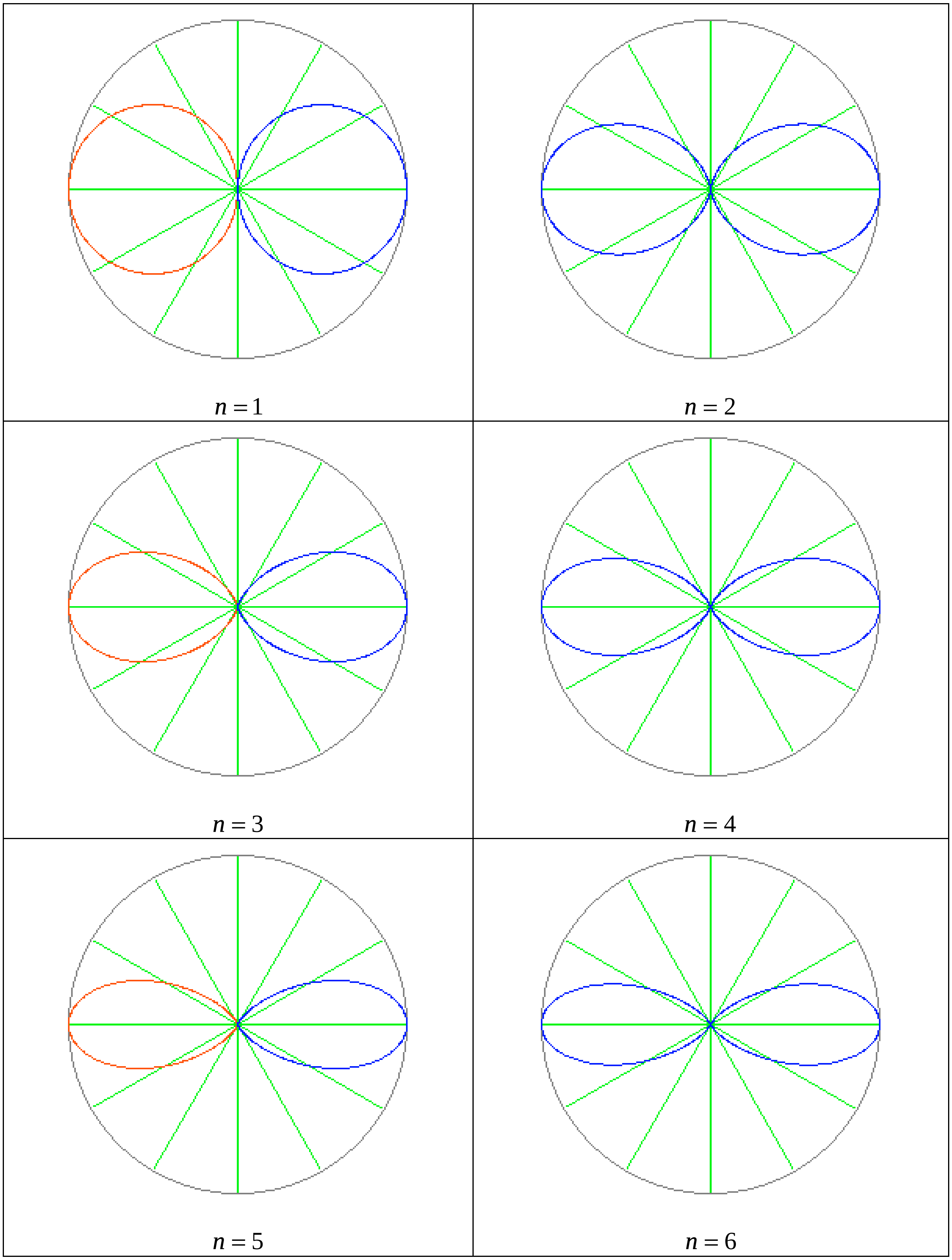
In passing, it may be observed that two of the polar response characteristics described above may easily be generalised to *n*th-order [70]. An *n*th-order figure-of-eight response is obtained by taking the *n*th-order directional derivative with all *n* direction vectors equal; the polar pattern is given by

$$M(q,f) = \cos^n(q)\cos^n(f) \tag{3.45}$$

The sum of figure-of-eight responses of order *n* and $n-1$ having a common axis and equal maximum responsivity is an *n*th-order cardioid for which

$$\begin{aligned} M(q,f) &= \frac{1}{2}\left[\cos^n(q)\cos^n(f) + \cos^{n-1}(q)\cos^{n-1}(f)\right] \\ &= \frac{1}{2}\left[1+\cos(q)\cos(f)\right]\cos^{n-1}(q)\cos^{n-1}(f) \end{aligned} \tag{3.46}$$

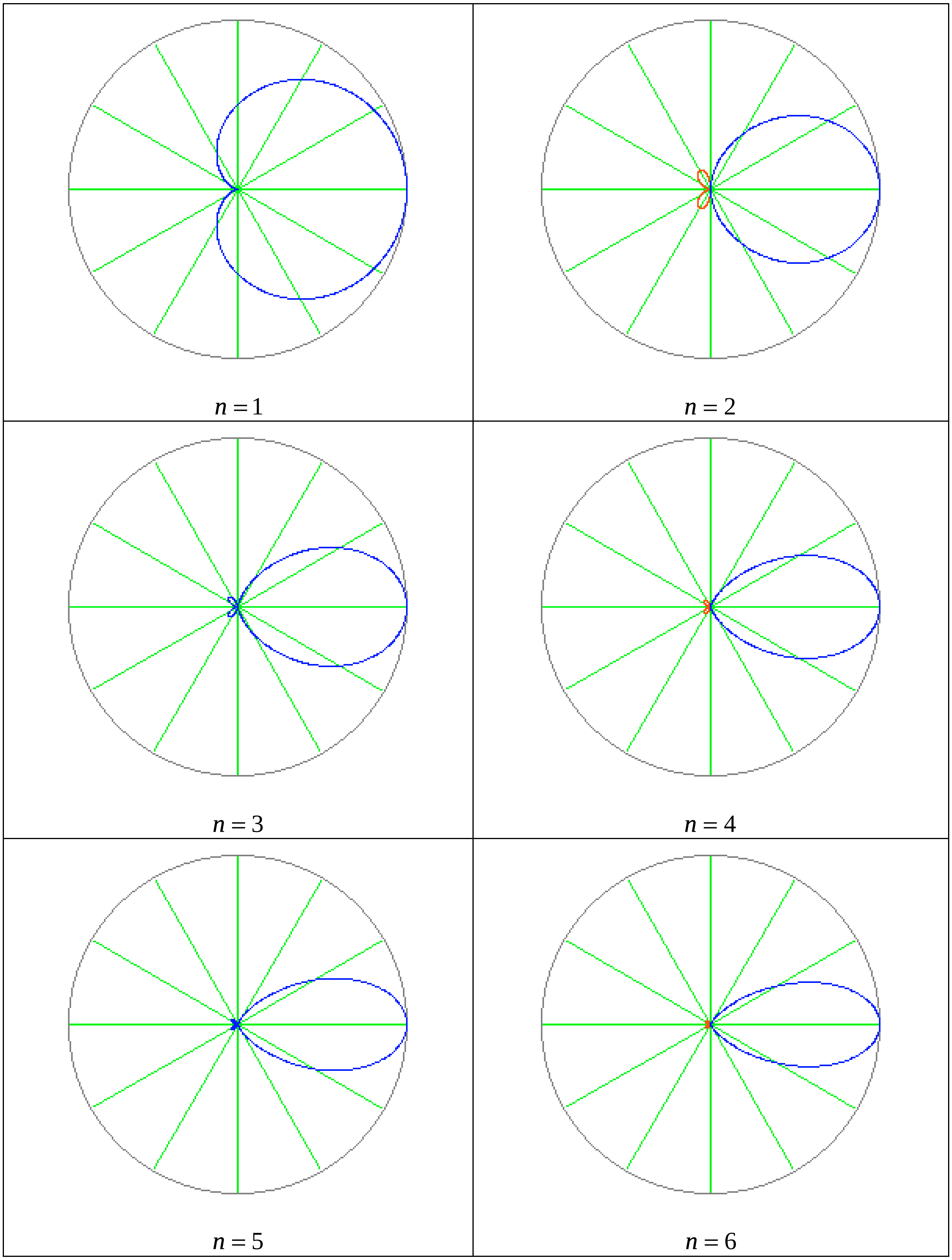
Plots of *n*th-order figure-of-eight and cardioid polar responses for values of *n* up to and including 6 are given in figures 3.5 and 3.6. These polar patterns all possess axial symmetry.



Blue → positive (in phase)

Red → negative (antiphase)

Figure 3.5: First-Order to Sixth-Order Figure-of-Eight Polar Patterns



Blue $\rightarrow$ positive (in phase)

Red $\rightarrow$ negative (antiphase)

Figure 3.6: First-Order to Sixth-Order Cardioid Polar Patterns

3.3: The Directivity Factor

One way in which the directional properties of a microphone may be quantified is by the "directivity factor", denoted by g (or sometimes by Q). This is defined as the ratio of the mean square output voltage generated by the microphone in response to a plane sound wave incident from the direction of maximum response to that generated in response to a perfectly uniform diffuse sound field having the same total acoustic power [11] [62]. Hence, the directivity factor measures the ability of the microphone to emphasise the desired sound source over ambient sound when that ambient sound is directionally homogeneous; this is often approximately the case, for example, for reverberant sound. Where there exist a small number of discrete, spatially localised sources of unwanted sound, the exact shape of the polar response, and particularly the locations of the response nulls, provides a more appropriate indicator of discriminatory ability.

The directivity factor is given by

$$g = \frac{4p}{\int_{0}^{2p}\int_{-p/2}^{p/2} \left|M(q,f)\right|^2 \cos(f)\,df\,dq} \qquad (3.47)$$

When the polar pattern exhibits axial symmetry, the directivity factor may found from the planar polar response using the simpler formula

$$g = \frac{2}{\int_{0}^{p} \left|M(q)\right|^2 \sin(q)\,dq} \qquad (3.48)$$

This applies to all first-order microphones; however, many second-order polar patterns are not axially symmetrical. Evaluating g for an arbitrary first-order microphone with polar pattern given by equation (3.17) gives the result

$$g = \frac{(a+b)^2}{a^2 + \frac{1}{3}b^2} \qquad (3.49)$$

It may be shown that the maximum possible directivity factor attainable using an ***n***th-order pressure gradient microphone is given by [43]

$$g_{\max}(n) = (n+1)^2 \qquad (3.50)$$

Hence, the maximum directivity factor which may be achieved using a first-order microphone is 4; this is in fact obtained with a 110°-null hypercardioid. First-order figure-of-eight and cardioid microphones both have directivity factors of 3 (and an omnidirectional microphone, by definition, has a directivity factor of 1). It is clear from equation (3.50) that second-order microphones may have directivity factors as high as 9; this is one reason why their use may be considered desirable, since it indicates a substantially greater ability to discriminate against ambient and reverberant sound [76].

3.4: Relationship Between Particle Velocity & Pressure Gradient

First-order pressure gradient microphones are often referred to as "velocity microphones"; in the opinion of the author this is unfortunate and potentially confusing, since it is the pressure gradient and not the particle velocity that actuates the microphone. The terminology can nevertheless be justified by reference to the relationship that exists between the pressure gradient and particle velocity fields in a sound wave.

As one of the fundamental equations governing the propagation of sound waves, we have [11] [68]

$$\nabla p = -r_0 \frac{\partial \mathbf{v}}{\partial t} \qquad (3.51)$$

where $\mathbf{v}$ is the instantaneous particle velocity and r_0 is the static (undisturbed) density of the medium; note that this equation does not depend on the sound field being a plane wave, but is of general applicability. We therefore have for the directional derivative of sound pressure

$$\hat{\mathbf{u}}_1 \cdot \nabla p = -r_0 \hat{\mathbf{u}}_1 \cdot \frac{\partial \mathbf{v}}{\partial t} \tag{3.52}$$

Integrating with respect to time and multiplying by *c*, we obtain

$$\begin{aligned} c\int (\hat{\mathbf{u}}_1 \cdot \nabla p)dt &= c\int \left(-r_0 \hat{\mathbf{u}}_1 \cdot \frac{\partial \mathbf{v}}{\partial t} \right) dt \\ &= -cr_0 \int \left(\hat{\mathbf{u}}_1 \cdot \frac{\partial \mathbf{v}}{\partial t} \right) dt \\ &= -cr_0 \hat{\mathbf{u}}_1 \cdot \int \left(\frac{\partial \mathbf{v}}{\partial t} \right) dt \\ &= -cr_0 \hat{\mathbf{u}}_1 \cdot \mathbf{v} \end{aligned} \tag{3.53}$$

The equalised pressure gradient is thus proportional to the component of particle velocity in the same direction, so that a pressure gradient microphone may be said to measure the particle velocity. The constant cr_0 is known as the “characteristic specific acoustic impedance” of the medium.

A possible misconception arising from use of the term “velocity microphone” is that a second-order pressure-gradient microphone measures the particle acceleration. This is not the case; since the particle acceleration is by definition the derivative with respect to time of the particle velocity, we may rearrange equation (3.52) to give

$$\hat{\mathbf{u}}_1 \cdot \frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{r_0} \hat{\mathbf{u}}_1 \cdot \nabla p \tag{3.54}$$

from which it is clear that the particle acceleration in a given direction is proportional to the first-order pressure gradient (without equalisation) in that direction. Meanwhile, taking the equalised directional derivative in the direction $\hat{\mathbf{u}}_2$ of each side of equation (3.53) gives

$$\begin{aligned} c^2 \int\int \left(\hat{\mathbf{u}}_2 \cdot \nabla(\hat{\mathbf{u}}_1 \cdot \nabla p) \right) dt dt &= c\int \hat{\mathbf{u}}_2 \cdot \nabla(-cr_0 \hat{\mathbf{u}}_1 \cdot \mathbf{v}) dt \\ &= -c^2 r_0 \int \hat{\mathbf{u}}_2 \cdot \nabla(\hat{\mathbf{u}}_1 \cdot \mathbf{v}) dt \\ &= -c^2 r_0 \hat{\mathbf{u}}_2 \cdot \nabla \left(\int (\hat{\mathbf{u}}_1 \cdot \mathbf{v}) dt \right) \\ &= -c^2 r_0 \hat{\mathbf{u}}_2 \cdot \nabla \left(\hat{\mathbf{u}}_1 \cdot \int \mathbf{v} dt \right) \end{aligned} \tag{3.55}$$

showing that an equalised second-order directional derivative of pressure is proportional to a first-order directional derivative of the component of particle displacement in a given direction.

3.5: The Proximity Effect

It is well known that the low frequency response of a microphone having a pressure gradient component is accentuated when the microphone is positioned close to a sound source. This is known as the "proximity effect", or (since it is the lower frequencies that are emphasised) the "bass boost effect"; it is generally considered to be undesirable, although in some musical applications the effect may be subjectively pleasing [13]. One motivation for seeking an understanding of this effect is that it facilitates the design of compensating filters. However, irrespective of whether such filter design is intended, the proximity effect is a significant factor in the performance of gradient microphones, and an understanding of it is required to properly characterise the expected behaviour of the second-order soundfield microphone.

An analysis of this effect will now be presented. Although some of the results regarding the first-order case are standard, the treatments presently available in the literature with which the author is familiar are somewhat unsatisfactory; even the more acceptable presentations (such as Beranek's [11]) lack generality and fail to adequately describe the derivation. Furthermore, many discussions are presented with reference only to a single type of transducer, so that the incorrect impression may sometimes be created that the proximity effect is a defect of a particular microphone design, whereas it is in fact an inevitable consequence of measuring the pressure gradient. A very few results relating to specific second-order microphones may be found in the literature [39] [76] [70]. These have again been given almost entirely without derivation; additionally, inconsistencies in the results presented in [39] suggest that they may not be entirely correct.

3.5.1: Proximity Effect for First-Order Microphones

Until now, the analysis presented has been concerned exclusively with plane waves. Close to a small source of sound, it is necessary instead to consider spherical waves [11] [89].

Assuming a point source located at the origin, the sound pressure field is described by

$$p = \frac{A'}{r} e^{j(wt - kr)} \tag{3.56}$$

where A' is the amplitude at unit distance from the source, and r is the distance from the source to the measurement point. The derivative with respect to r

$$\begin{aligned} \frac{\partial p}{\partial r} &= -A' \left(\frac{1 + jkr}{r^2} \right) e^{j(wt - kr)} \\ &= -\left(\frac{1 + jkr}{r} \right) p \end{aligned} \tag{3.57}$$

In spherical polar coordinates, the gradient of a function f is given by [68]

$$\nabla f = \begin{bmatrix} \dfrac{\partial f}{\partial r} \\ \dfrac{1}{r \cos(f)} \dfrac{\partial f}{\partial q} \\ \dfrac{1}{r} \dfrac{\partial f}{\partial f} \end{bmatrix} \tag{3.58}$$

so that

$$\begin{aligned} \nabla p &= \begin{bmatrix} -A' \left(\dfrac{1 + jkr}{r^2} \right) e^{j(wt - kr)} \\ 0 \\ 0 \end{bmatrix} \\ &= -A' \left(\frac{1 + jkr}{r^2} \right) e^{j(wt - kr)} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ &= -A' \left(\frac{1 + jkr}{r^2} \right) e^{j(wt - kr)} \hat{\mathbf{u}}_r \\ &= -\left(\frac{1 + jkr}{r} \right) p \hat{\mathbf{u}}_r \end{aligned} \tag{3.59}$$

where $\hat{\mathbf{u}}_r$ is a unit vector in the radial direction; that is, pointing directly away from the origin. It is convenient now to convert this expression into cartesian coordinates. At the point $(r,\bar{q},\bar{f})$,

$$\hat{\mathbf{u}}_r = \cos(\bar{q})\cos(\bar{f})\hat{\mathbf{x}} + \sin(\bar{q})\cos(\bar{f})\hat{\mathbf{y}} + \sin(\bar{f})\hat{\mathbf{z}} \tag{3.60}$$

(see figure 3.7).

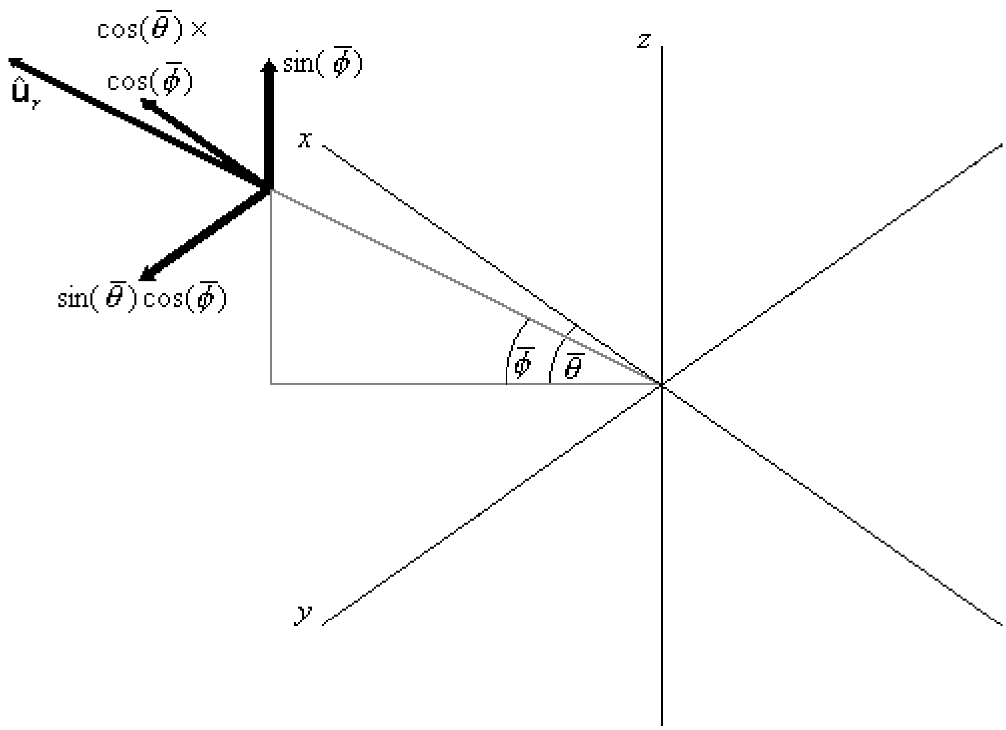


Figure 3.7: Cartesian Components of Radial Vector $\hat{\mathbf{u}}_r$

Substituting for $\hat{\mathbf{u}}_r$ in equation (3.59) gives

$$\nabla p = -A'\left(\frac{1+jkr}{r^2}\right)e^{j(wt-kr)}\begin{bmatrix}\cos(\bar{q})\cos(\bar{f})\\ \sin(\bar{q})\cos(\bar{f})\\ \sin(\bar{f})\end{bmatrix} \tag{3.61}$$

It is clear from this that ∇p depends on the point at which the measurement is taken. This must of course remain true whatever coordinate system the gradient is expressed in, but as can be seen from equation (3.59), the dependency does not appear explicitly when the result is written in polar coordinates; rather, it is implicit in the use of $\hat{\mathbf{u}}_r$, which itself depends on position, as is evident from equation (3.60). Now, as before, let $\hat{\mathbf{u}}_1$ be a unit vector pointing in a direction (q', f');

$$\hat{\mathbf{u}}_1 = \begin{bmatrix} \cos(q')\cos(f') \\ \sin(q')\cos(f') \\ \sin(f') \end{bmatrix} \tag{3.62}$$

The directional derivative in the direction $\hat{\mathbf{u}}_1$ is then given by

$$\begin{aligned} \hat{\mathbf{u}}_1 \cdot \nabla p &= -A'\left(\frac{1+jkr}{r^2}\right)\left[\cos(\bar{f})\cos(f')\cos(\bar{q}-q') + \sin(\bar{f})\sin(f')\right]e^{j(wt-kr)} \\ &= -\left(\frac{1+jkr}{r}\right)\left[\cos(\bar{f})\cos(f')\cos(\bar{q}-q') + \sin(\bar{f})\sin(f')\right]p \end{aligned} \tag{3.63}$$

which is similar to equation (3.6), although the complex "gain" is a more complicated function in this case. It should however be remembered that the propagation direction $(\bar{q}, \bar{f})$ depends on the point at which the gradient is taken, whereas in the case of a plane wave the direction of propagation is everywhere the same.

It is again useful to express this result in term of a coordinate system centred on the microphone. Consider a point source located a distance *r* from the origin, with coordinates (x_s, y_s, z_s); then

$$r = \sqrt{x_s^{\,2} + y_s^{\,2} + z_s^{\,2}} \tag{3.64}$$

Let $\tilde{r}$ be the distance from the source to an arbitrary point (x, y, z):

$$\tilde{r} = \sqrt{(x-x_s)^2 + (y-y_s)^2 + (z-z_s)^2} \tag{3.65}$$

The pressure at such a point is then given by

$$p = \frac{A'}{\tilde{r}} e^{j(wt - k\tilde{r})} \tag{3.66}$$

and, by the chain rule,

$$\begin{aligned} \nabla p &= \frac{\partial}{\partial \tilde{r}} \left\{ \frac{A'}{\tilde{r}} e^{j(wt - k\tilde{r})} \right\} \times \begin{bmatrix} \frac{\partial \tilde{r}}{\partial x} \\ \frac{\partial \tilde{r}}{\partial y} \\ \frac{\partial \tilde{r}}{\partial z} \end{bmatrix} \\ &= -A' \left(\frac{1 + jk\tilde{r}}{\tilde{r}^2} \right) e^{j(wt - k\tilde{r})} \frac{1}{\tilde{r}} \begin{bmatrix} x - x_s \\ y - y_s \\ z - z_s \end{bmatrix} \end{aligned} \tag{3.67}$$

Since the coordinate system is centred on the microphone, we evaluate this at the origin, so that $x = y = z = 0$ and $\tilde{r} = r$;

$$\begin{aligned} \nabla p|_O &= -A' \left(\frac{1 + jkr}{r^2} \right) e^{j(wt - kr)} \frac{1}{r} \begin{bmatrix} -x_s \\ -y_s \\ -z_s \end{bmatrix} \\ &= A' \left(\frac{1 + jkr}{r^2} \right) e^{j(wt - kr)} \hat{\mathbf{r}} \end{aligned} \tag{3.68}$$

where $\hat{\mathbf{r}}$ is a unit vector pointing in the direction of the source; hence, if that direction is defined by angles (q, f), then

$$\hat{\mathbf{r}} = \begin{bmatrix} \cos(q)\cos(f) \\ \sin(q)\cos(f) \\ \sin(f) \end{bmatrix} \tag{3.69}$$

Adopting as before the convention that $\hat{\mathbf{u}}_1$ is aligned with the x axis of the microphone coordinate system, the pressure gradient is given by

$$\hat{\mathbf{x}}\cdot\nabla p\Big|_O = A'\cos(q)\cos(f)\left(\frac{1+jkr}{r^2}\right)e^{j(wt-kr)} \tag{3.70}$$

and the equalised pressure gradient by

$$\begin{aligned} c\int\left(\hat{\mathbf{x}}\cdot\nabla p\Big|_O\right)dt &= A'\cos(q)\cos(f)\left(\frac{1+jkr}{jkr^2}\right)e^{j(wt-kr)} \\ &= \cos(q)\cos(f)\left(\frac{1+jkr}{jkr}\right)p \end{aligned} \tag{3.71}$$

so that the output of a first-order pressure gradient microphone in response to a spherical wave is

$$\begin{aligned} v_{grad} &= GA'\cos(q)\cos(f)\left(\frac{1+jkr}{jkr^2}\right)e^{j(wt-kr)} \\ &= G\cos(q)\cos(f)\left(\frac{1+jkr}{jkr}\right)p \end{aligned} \tag{3.72}$$

The proximity effect bass boost factor is defined as the ratio of the amplitude of the pressure gradient microphone output to the amplitude of the output of an omnidirectional pressure microphone, when both are positioned at the same point in the sound field, and the pressure microphone is such that its response to a plane wave is identical to the response of the gradient microphone to a plane wave arriving on axis. The output of the pressure microphone in this case is therefore

$$v_{omni} = G\frac{A'}{r}e^{j(wt-kr)} \tag{3.73}$$

It should be noted that there exist slightly different definitions of the bass boost factor, which depend on using a pressure microphone with different responsivity as the basis for comparison; in most cases, the results obtained using alternative definitions (such as those quoted in [76]) differ only by a multiplying constant. Let the bass boost factor be denoted *B*, then

$$\begin{aligned} B &= \frac{|v_{grad}|}{|v_{omni}|} \\ &= \frac{GA'|\cos(q)\cos(f)|\left|\dfrac{1+jkr}{jkr^2}\right|}{G\dfrac{A'}{r}} \\ &= |\cos(q)\cos(f)|\,r\,\frac{\sqrt{1+k^2r^2}}{kr^2} \\ &= |\cos(q)\cos(f)|\frac{\sqrt{1+k^2r^2}}{kr} \end{aligned} \qquad (3.74)$$

Consider the case of on-axis incidence, which is the most practically important case (and the one most often mentioned in the literature); $\cos(q)\cos(f) = 1$ and

$$B = \frac{\sqrt{1+k^2r^2}}{kr} \qquad (3.75)$$

When *r* is large compared to the wavelength of the incident sound wave, *kr* is large compared to unity and

$$B \approx \frac{\sqrt{k^2r^2}}{kr} = 1 \qquad (3.76)$$

Therefore, at distances from the source which are large compared to the wavelength, there is no bass boost. This is to be expected, since at a distance from a small source which is large compared to wavelength the spherical wavefront may be approximated over a small region by a plane wavefront.

When *r* is small compared to wavelength, *kr* is small compared to unity and

$$B \approx \frac{\sqrt{1}}{kr} = \frac{1}{kr} \qquad (3.77)$$

Hence, close to the source the bass boost varies inversely with *kr*; at a fixed distance, the microphone output rises by 6 dB for each octave drop in frequency. This is a well-known result [13] [39] [76] [70].

For other directions of incidence, the response varies with *kr* in the same way, although the bass boost is scaled in accordance with the polar pattern. This scaling is, of course, a consequence of the decision to compare with a pressure microphone having a plane wave response equal to the maximum response of the gradient microphone. Note that for $q = \pm 90°$, $B = 0$, because the gradient microphone does not respond to waves from any direction in that plane.

We now consider the general case of a microphone having a mixed pressure and pressure gradient response. Substituting equations (3.66) (with $\tilde{r} = r$) and (3.71) into equation (3.15) gives a microphone output

$$\begin{aligned} v &= G\frac{1}{a+b}\left[a\frac{A'}{r}e^{j(wt-kr)} + bA'\cos(q)\cos(f)\left(\frac{1+jkr}{jkr^2}\right)e^{j(wt-kr)}\right] \\ &= GA'\frac{1}{a+b}\frac{1}{r}\left[a + b\cos(q)\cos(f)\left(\frac{1+jkr}{jkr}\right)\right]e^{j(wt-kr)} \\ &= G\frac{1}{a+b}\left[a + b\cos(q)\cos(f)\left(\frac{1+jkr}{jkr}\right)\right]p \end{aligned} \tag{3.78}$$

so that

$$\begin{aligned} B &= \frac{1}{a+b}\left|a + b\cos(q)\cos(f)\left(\frac{1+jkr}{jkr}\right)\right| \\ &= \frac{1}{a+b}\left|\frac{jakr}{jkr} + \frac{b\cos(q)\cos(f)}{jkr} + \frac{jbkr\cos(q)\cos(f)}{jkr}\right| \\ &= \frac{1}{a+b}\left|\frac{b\cos(q)\cos(f) + jkr(a + b\cos(q)\cos(f))}{jkr}\right| \\ &= \frac{1}{a+b}\frac{\sqrt{b^2\cos^2(q)\cos^2(f) + k^2r^2(a + b\cos(q)\cos(f))^2}}{kr} \end{aligned} \tag{3.79}$$

This allows the bass boost to be calculated for any first-order microphone and any direction of incidence. Consider a cardioid, with $a = b = 1$:

$$B = \frac{1}{2}\frac{\sqrt{\cos^2(q)\cos^2(f) + k^2r^2(1+\cos(q)\cos(f))^2}}{kr} \qquad (3.80)$$

For frontal incidence, this gives

$$B = \frac{1}{2}\frac{\sqrt{1+4k^2r^2}}{kr} \qquad (3.81)$$

which is not dissimilar to equation (3.75). For large *kr* this again tends to unity, as one would expect. For small *kr*

$$B \approx \frac{1}{2kr} \qquad (3.82)$$

This again gives a 6 dB rise for every halving of *kr*, but 6 dB below the response of the pure pressure gradient microphone. This is another standard result; the lower boost occurs because only the pressure gradient element of the response is accentuated by the proximity effect, and so the proportional increase in the total output is less. For incidence from the rear ($q = 180°$ and $f = 0°$, so that $\cos(q)\cos(f) = -1$),

$$B = \frac{1}{2kr} \qquad (3.83)$$

The cardioid pattern has a null in this position, and for large *kr*, *B* becomes negligibly small, approaching the zero response that would be produced by a plane wave. However, for close sources, the implication is that the response no longer shows a null in this position. This is a reasonable conclusion; the null is due to cancellation between the pressure and pressure gradient components, and depends on them being equal in magnitude and in exact antiphase. Close to a source of spherical waves, the pressure gradient is increased in amplitude and phase-shifted, so exact cancellation does not occur. This reasoning is also applicable to other composite microphones; hence, it may be concluded that the polar pattern of a microphone which responds to a combination of pressure and pressure gradient will change when the microphone is placed close to a small sound source.

3.5.2: Proximity Effect for Second-Order Microphones

We now proceed to consider the response of second-order gradient microphones to spherical waves. From equation (3.67), we know that

$$\frac{\partial p}{\partial x} = -\frac{A'}{\tilde{r}}(x - x_s)\left(\frac{1 + jk\tilde{r}}{\tilde{r}^2}\right)e^{j(wt - k\tilde{r})} \tag{3.84a}$$

$$\frac{\partial p}{\partial y} = -\frac{A'}{\tilde{r}}(y - y_s)\left(\frac{1 + jk\tilde{r}}{\tilde{r}^2}\right)e^{j(wt - k\tilde{r})} \tag{3.84b}$$

$$\frac{\partial p}{\partial z} = -\frac{A'}{\tilde{r}}(z - z_s)\left(\frac{1 + jk\tilde{r}}{\tilde{r}^2}\right)e^{j(wt - k\tilde{r})} \tag{3.84c}$$

Differentiating a second time gives

$$\frac{\partial^2 p}{\partial x^2} = -\frac{A'}{\tilde{r}}\left[\frac{1 + jk\tilde{r}}{\tilde{r}^2} + (x - x_s)^2\left(\frac{-3 - j3k\tilde{r} + k^2\tilde{r}^2}{\tilde{r}^4}\right)\right]e^{j(wt - k\tilde{r})} \tag{3.85a}$$

$$\frac{\partial^2 p}{\partial y^2} = -\frac{A'}{\tilde{r}}\left[\frac{1 + jk\tilde{r}}{\tilde{r}^2} + (y - y_s)^2\left(\frac{-3 - j3k\tilde{r} + k^2\tilde{r}^2}{\tilde{r}^4}\right)\right]e^{j(wt - k\tilde{r})} \tag{3.85b}$$

$$\frac{\partial^2 p}{\partial z^2} = -\frac{A'}{\tilde{r}}\left[\frac{1 + jk\tilde{r}}{\tilde{r}^2} + (z - z_s)^2\left(\frac{-3 - j3k\tilde{r} + k^2\tilde{r}^2}{\tilde{r}^4}\right)\right]e^{j(wt - k\tilde{r})} \tag{3.85c}$$

$$\frac{\partial^2 p}{\partial x \partial y} = -\frac{A'}{\tilde{r}}(x - x_s)(y - y_s)\left(\frac{-3 - j3k\tilde{r} + k^2\tilde{r}^2}{\tilde{r}^4}\right)e^{j(wt - k\tilde{r})} \tag{3.85d}$$

$$\frac{\partial^2 p}{\partial x \partial z} = -\frac{A'}{\tilde{r}}(x - x_s)(z - z_s)\left(\frac{-3 - j3k\tilde{r} + k^2\tilde{r}^2}{\tilde{r}^4}\right)e^{j(wt - k\tilde{r})} \tag{3.85e}$$

$$\frac{\partial^2 p}{\partial y \partial z} = -\frac{A'}{\tilde{r}}(y - y_s)(z - z_s)\left(\frac{-3 - j3k\tilde{r} + k^2\tilde{r}^2}{\tilde{r}^4}\right)e^{j(wt - k\tilde{r})} \tag{3.85f}$$

As previously, we now assume that the measurement point is the origin, so that $x = y = z = 0$, $\tilde{r} = r$, and

$$\frac{\partial^2 p}{\partial x^2} = -\frac{A'}{r}\left[\frac{1 + jkr}{r^2} + x_s{}^2\left(\frac{-3 - j3kr + k^2r^2}{r^4}\right)\right]e^{j(wt - kr)} \tag{3.86a}$$

$$\frac{\partial^2 p}{\partial y^2} = -\frac{A'}{r}\left[\frac{1 + jkr}{r^2} + y_s{}^2\left(\frac{-3 - j3kr + k^2r^2}{r^4}\right)\right]e^{j(wt - kr)} \tag{3.86b}$$

$$\frac{\partial^2 p}{\partial z^2} = -\frac{A'}{r}\left[\frac{1+jkr}{r^2} + {z_s}^2\left(\frac{-3-j3kr+k^2r^2}{r^4}\right)\right]e^{j(wt-kr)} \tag{3.86c}$$

$$\frac{\partial^2 p}{\partial x \partial y} = -\frac{A'}{r}x_s y_s\left(\frac{-3-j3kr+k^2r^2}{r^4}\right)e^{j(wt-kr)} \tag{3.86d}$$

$$\frac{\partial^2 p}{\partial x \partial z} = -\frac{A'}{r}x_s z_s\left(\frac{-3-j3kr+k^2r^2}{r^4}\right)e^{j(wt-kr)} \tag{3.86e}$$

$$\frac{\partial^2 p}{\partial y \partial z} = -\frac{A'}{r}y_s z_s\left(\frac{-3-j3kr+k^2r^2}{r^4}\right)e^{j(wt-kr)} \tag{3.86f}$$

Expressing the source location in spherical polar coordinate form, we obtain

$$\frac{\partial^2 p}{\partial x^2} = -\frac{A'}{r}\left[\frac{1+jkr}{r^2} + \cos^2(q)\cos^2(f)\left(\frac{-3-j3kr+k^2r^2}{r^2}\right)\right]e^{j(wt-kr)} \tag{3.87a}$$

$$\frac{\partial^2 p}{\partial y^2} = -\frac{A'}{r}\left[\frac{1+jkr}{r^2} + \sin^2(q)\cos^2(f)\left(\frac{-3-j3kr+k^2r^2}{r^2}\right)\right]e^{j(wt-kr)} \tag{3.87b}$$

$$\frac{\partial^2 p}{\partial z^2} = -\frac{A'}{r}\left[\frac{1+jkr}{r^2} + \sin^2(f)\left(\frac{-3-j3kr+k^2r^2}{r^2}\right)\right]e^{j(wt-kr)} \tag{3.87c}$$

$$\frac{\partial^2 p}{\partial x \partial y} = -\frac{A'}{r}\cos(q)\sin(q)\cos^2(f)\left(\frac{-3-j3kr+k^2r^2}{r^2}\right)e^{j(wt-kr)} \tag{3.87d}$$

$$\frac{\partial^2 p}{\partial x \partial z} = -\frac{A'}{r}\cos(q)\cos(f)\sin(f)\left(\frac{-3-j3kr+k^2r^2}{r^2}\right)e^{j(wt-kr)} \tag{3.87e}$$

$$\frac{\partial^2 p}{\partial y \partial z} = -\frac{A'}{r}\sin(q)\cos(f)\sin(f)\left(\frac{-3-j3kr+k^2r^2}{r^2}\right)e^{j(wt-kr)} \tag{3.87f}$$

Finally, application of double-integration filtering gives

$$\begin{aligned} c^2\iint\frac{\partial^2 p}{\partial x^2}dtdt &= \frac{A'}{r}\left[\frac{1+jkr}{k^2r^2} + \cos^2(q)\cos^2(f)\left(\frac{-3-j3kr+k^2r^2}{k^2r^2}\right)\right]e^{j(wt-kr)} \\ &= \left[\frac{1+jkr}{k^2r^2} + \cos^2(q)\cos^2(f)\left(\frac{-3-j3kr+k^2r^2}{k^2r^2}\right)\right]p \end{aligned} \tag{3.88a}$$

$$\begin{aligned} c^2\iint\frac{\partial^2 p}{\partial y^2}dtdt &= \frac{A'}{r}\left[\frac{1+jkr}{k^2r^2} + \sin^2(q)\cos^2(f)\left(\frac{-3-j3kr+k^2r^2}{k^2r^2}\right)\right]e^{j(wt-kr)} \\ &= \left[\frac{1+jkr}{k^2r^2} + \sin^2(q)\cos^2(f)\left(\frac{-3-j3kr+k^2r^2}{k^2r^2}\right)\right]p \end{aligned} \tag{3.88b}$$

$$c^2\iint\frac{\partial^2 p}{\partial z^2}dtdt = \frac{A'}{r}\left[\frac{1+jkr}{k^2r^2}+\sin^2(f)\left(\frac{-3-j3kr+k^2r^2}{k^2r^2}\right)\right]e^{j(wt-kr)}$$
$$=\left[\frac{1+jkr}{k^2r^2}+\sin^2(f)\left(\frac{-3-j3kr+k^2r^2}{k^2r^2}\right)\right]p \tag{3.88c}$$

$$c^2\iint\frac{\partial^2 p}{\partial x\partial y}dtdt = \frac{A'}{r}\cos(q)\sin(q)\cos^2(f)\left(\frac{-3-j3kr+k^2r^2}{k^2r^2}\right)e^{j(wt-kr)}$$
$$=\cos(q)\sin(q)\cos^2(f)\left(\frac{-3-j3kr+k^2r^2}{k^2r^2}\right)p \tag{3.88d}$$

$$c^2\iint\frac{\partial^2 p}{\partial x\partial z}dtdt = \frac{A'}{r}\cos(q)\cos(f)\sin(f)\left(\frac{-3-j3kr+k^2r^2}{k^2r^2}\right)e^{j(wt-kr)}$$
$$=\cos(q)\cos(f)\sin(f)\left(\frac{-3-j3kr+k^2r^2}{k^2r^2}\right)p \tag{3.88e}$$

$$c^2\iint\frac{\partial^2 p}{\partial y\partial z}dtdt = \frac{A'}{r}\sin(q)\cos(f)\sin(f)\left(\frac{-3-j3kr+k^2r^2}{k^2r^2}\right)e^{j(wt-kr)}$$
$$=\sin(q)\cos(f)\sin(f)\left(\frac{-3-j3kr+k^2r^2}{k^2r^2}\right)p \tag{3.88f}$$

To establish the response to a spherical wave of an arbitrary second-order gradient microphone, we consider the equalised version of equation (3.34):

$$\begin{aligned}
c^2\iint\left(\hat{\mathbf{u}}_2\cdot\nabla(\hat{\mathbf{u}}_1\cdot\nabla p)\right)dtdt = c^2\iint\Bigg[& x_1x_2\frac{\partial^2 p}{\partial x^2}+(x_1y_2+y_1x_2)\frac{\partial^2 p}{\partial x\partial y} \\
&+(x_1z_2+z_1x_2)\frac{\partial^2 p}{\partial x\partial z}+y_1y_2\frac{\partial^2 p}{\partial y^2} \\
&+(y_1z_2+z_1y_2)\frac{\partial^2 p}{\partial y\partial z}+z_1z_2\frac{\partial^2 p}{\partial z^2}\Bigg]dtdt \\
= x_1x_2c^2\iint\frac{\partial^2 p}{\partial x^2}dtdt &+ (x_1y_2+y_1x_2)c^2\iint\frac{\partial^2 p}{\partial x\partial y}dtdt \\
+(x_1z_2+z_1x_2)c^2\iint\frac{\partial^2 p}{\partial x\partial z}dtdt &+ y_1y_2c^2\iint\frac{\partial^2 p}{\partial y^2}dtdt \\
+(y_1z_2+z_1y_2)c^2\iint\frac{\partial^2 p}{\partial y\partial z}dtdt &+ z_1z_2c^2\iint\frac{\partial^2 p}{\partial z^2}dtdt
\end{aligned} \tag{3.89}$$

Substituting in the equalised partial derivatives from equation (3.88), we obtain

$$c^2 \iint \left(\hat{\mathbf{u}}_2 \cdot \nabla(\hat{\mathbf{u}}_1 \cdot \nabla p)\right) dt dt = M(q,f)\, p \tag{3.90}$$

where

$$\begin{aligned} M(q,f) = {} & C_1\left[\frac{1+jkr}{k^2r^2} + \cos^2(q)\cos^2(f)\left(\frac{-3-j3kr+k^2r^2}{k^2r^2}\right)\right] \\ & + C_2\cos(q)\sin(q)\cos^2(f)\left(\frac{-3-j3kr+k^2r^2}{k^2r^2}\right) \\ & + C_3\cos(q)\cos(f)\sin(f)\left(\frac{-3-j3kr+k^2r^2}{k^2r^2}\right) \\ & + C_4\left[\frac{1+jkr}{k^2r^2} + \sin^2(q)\cos^2(f)\left(\frac{-3-j3kr+k^2r^2}{k^2r^2}\right)\right] \\ & + C_5\sin(q)\cos(f)\sin(f)\left(\frac{-3-j3kr+k^2r^2}{k^2r^2}\right) \\ & + C_6\left[\frac{1+jkr}{k^2r^2} + \sin^2(f)\left(\frac{-3-j3kr+k^2r^2}{k^2r^2}\right)\right] \end{aligned} \tag{3.91}$$

with C_1, C_2, etc., defined as before by equation (3.38). The Laplace series for $M(q,f)$ in this case has coefficients (obtained as before by using the formulæ given in Chapter 2 or by trigonometric and algebraic manipulation)

$$A_0 = \tfrac{1}{3}C_1 + \tfrac{1}{3}C_4 + \tfrac{1}{3}C_6 \tag{3.92a}$$

$$A_1 = 0 \tag{3.92b}$$

$$A_2 = \left[\tfrac{2}{3}C_6 - \tfrac{1}{3}C_1 - \tfrac{1}{3}C_4\right]\left(\frac{-3-j3kr+k^2r^2}{k^2r^2}\right) \tag{3.92c}$$

$$A_{1,1} = 0 \tag{3.92d}$$

$$A_{2,1} = \tfrac{1}{3}C_3\left(\frac{-3-j3kr+k^2r^2}{k^2r^2}\right) \tag{3.92e}$$

$$A_{2,2} = \left[\tfrac{1}{6}C_1 - \tfrac{1}{6}C_4\right]\left(\frac{-3-j3kr+k^2r^2}{k^2r^2}\right) \tag{3.92f}$$

$$B_{1,1} = 0 \tag{3.92g}$$

$$B_{2,1} = \tfrac{1}{3}C_5\left(\frac{-3-j3kr+k^2r^2}{k^2r^2}\right) \tag{3.92h}$$

$$B_{2,2} = \tfrac{1}{6} C_2 \left(\frac{-3 - j3kr + k^2 r^2}{k^2 r^2} \right) \tag{3.92i}$$

Comparing with equation (3.39), we see that these coefficients have the same values as in the plane wave case, except that every second-order coefficient is multiplied by the factor $(-3 - j3kr + k^2 r^2)/k^2 r^2$. The response of a second-order gradient microphone to a spherical wave is therefore closely related to its plane wave response, and indeed can be found using the Laplace series decomposition of its polar pattern; from equation (3.90), including an overall responsivity constant *G*, we may write for the output of a second-order pressure gradient microphone in response to a spherical wave

$$v = G\left[A_0 + M_2(q,f)\left(\frac{-3 - j3kr + k^2 r^2}{k^2 r^2} \right)\right] p \tag{3.93}$$

where $M_2(q,f)$ is the second-order spherical harmonic component of the polar pattern; i.e.,

$$\begin{aligned} M_2(q,f) = \tfrac{1}{2} A_2[3\sin^2(f) - 1] + 3A_{2,1}\cos(q)\cos(f)\sin(f) + 3A_{2,2}\cos(2q)\cos^2(f) \\ +3B_{2,1}\sin(q)\cos(f)\sin(f) + 3B_{2,2}\sin(2q)\cos^2(f) \end{aligned} \tag{3.94}$$

From this, we obtain for the bass boost factor

$$B = \left| A_0 + M_2(q,f)\left(\frac{-3 - j3kr + k^2 r^2}{k^2 r^2} \right) \right| \tag{3.95}$$

A consequence of this result is that the tesseral quadrupole is the only second-order pressure gradient microphone for which the polar pattern is independent of *kr* in the spherical wave case, since every other second-order polar pattern includes an omnidirectional component.

By way of example, consider the second-order figure-of-eight response described previously. The Laplace series expansion of the polar pattern in this case has three non-zero coefficients:

$$A_0 = \tfrac{1}{3} \tag{3.96a}$$

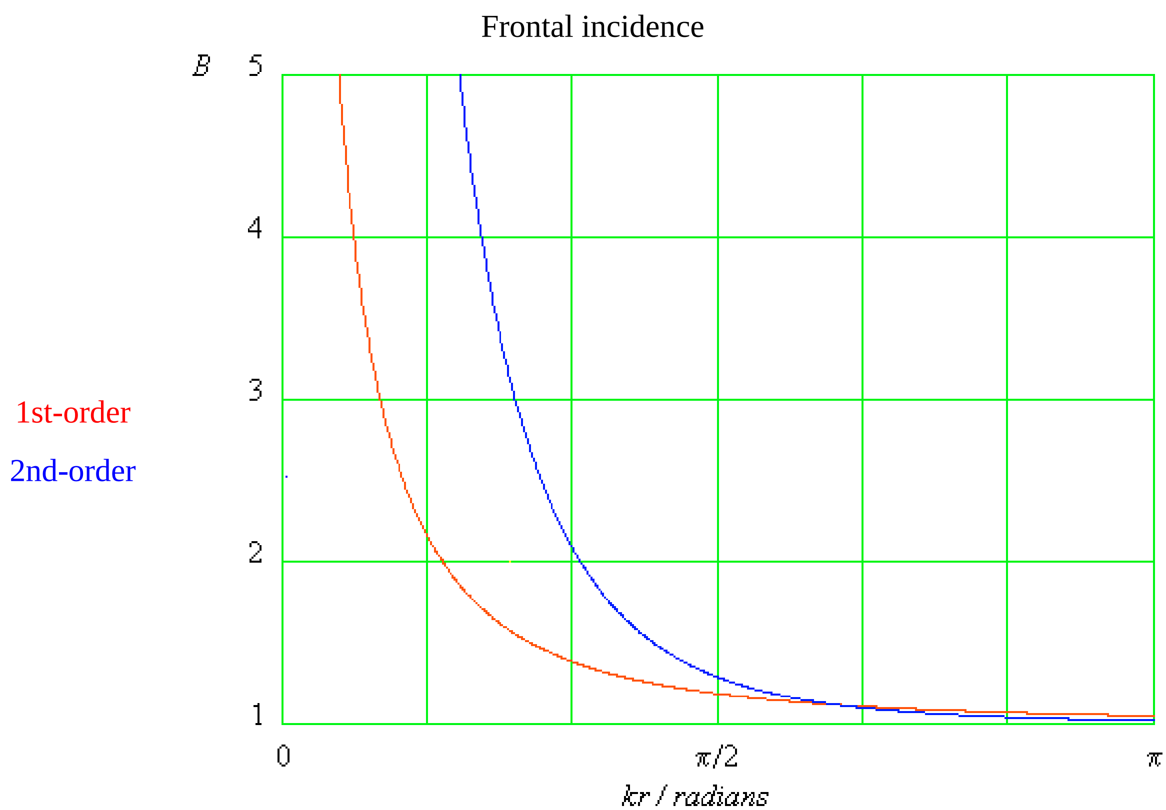
$$A_2 = -\tfrac{1}{3} \tag{3.96b}$$

$$A_{2,2} = \tfrac{1}{6} \tag{3.96c}$$

Therefore, the bass boost factor

$$\begin{aligned}
B &= \left|\frac{1}{3} + \left(-\frac{1}{3}\frac{1}{2}[3\sin^2(f) - 1] + \frac{1}{6}3\cos(2q)\cos^2(f)\right)\left(\frac{-3 - j3kr + k^2r^2}{k^2r^2}\right)\right| \\
&= \left|\frac{1}{3} + \left(-\frac{1}{6}[3\sin^2(f) - 1] + \frac{1}{2}\cos(2q)\cos^2(f)\right)\left(\frac{-3 - j3kr + k^2r^2}{k^2r^2}\right)\right| \\
&= \left|\frac{1}{3} + \left(-\frac{1}{2}\sin^2(f) + \frac{1}{6} + \frac{1}{2}\cos^2(q)\cos^2(f) - \frac{1}{2}\sin^2(q)\cos^2(f)\right)\right. \\
&\qquad \left. \times\left(\frac{-3 - j3kr + k^2r^2}{k^2r^2}\right)\right| \\
&= \left|\frac{1}{3} + \left(-\frac{1}{2} + \frac{1}{2}\cos^2(f) + \frac{1}{6} + \frac{1}{2}\cos^2(q)\cos^2(f) - \frac{1}{2}\cos^2(f)\right.\right. \\
&\qquad \left.\left. + \frac{1}{2}\cos^2(q)\cos^2(f)\right)\left(\frac{-3 - j3kr + k^2r^2}{k^2r^2}\right)\right| \\
&= \left|\frac{1}{3} + \left(-\frac{1}{3} + \cos^2(q)\cos^2(f)\right)\left(\frac{-3 - j3kr + k^2r^2}{k^2r^2}\right)\right| \\
&= \left|\frac{1}{3}\frac{k^2r^2}{k^2r^2} - \frac{1}{3}\left(\frac{-3 - j3kr + k^2r^2}{k^2r^2}\right) + \cos^2(q)\cos^2(f)\left(\frac{-3 - j3kr + k^2r^2}{k^2r^2}\right)\right| \qquad (3.97) \\
&= \left|\frac{1 + jkr}{k^2r^2} + \cos^2(q)\cos^2(f)\left(\frac{-3 - j3kr + k^2r^2}{k^2r^2}\right)\right|
\end{aligned}$$

This can be seen to be the result that would be obtained using equation (3.88a) directly to find the response of the microphone. Graphs of B for frontal and 90° off-axis incidence, for both first-order and second-order pressure gradient microphones, are given in figure 3.8.



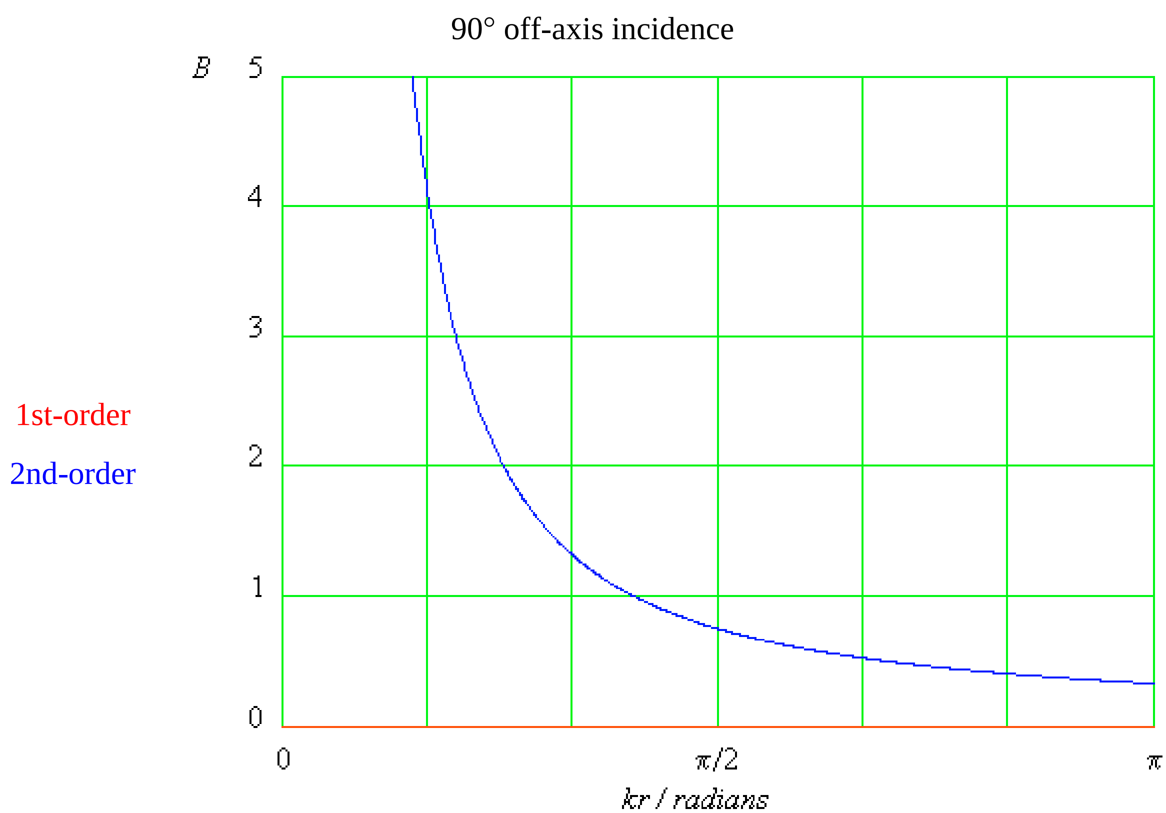


Figure 3.8: Bass Boost for First-Order and Second-Order Pressure Gradient Microphones

If first-order components are also present, so that we have a microphone of the type described by equation (3.30), then the response to a spherical wave is

$$v = G\left[A_0 + M_1(q,f)\left(\frac{1+jkr}{jkr}\right) + M_2(q,f)\left(\frac{-3-j3kr+k^2r^2}{k^2r^2}\right)\right]p \tag{3.98}$$

and the bass boost factor is

$$B = \left|A_0 + M_1(q,f)\left(\frac{1+jkr}{jkr}\right) + M_2(q,f)\left(\frac{-3-j3kr+k^2r^2}{k^2r^2}\right)\right| \tag{3.99}$$

where $M_1(q,f)$ is the first-order harmonic component of the polar response and $M_2(q,f)$ is, as previously, the second-order harmonic component. Note that direct measurement of pressure, as well as the second-order derivatives, may contribute to the zeroth-order component.

An example of a microphone having spherical harmonic components of all three orders is the second-order cardioid described by equation (3.31). The Laplace series in this case has coefficients

$$A_0 = \tfrac{1}{6} \tag{3.100a}$$

$$A_{1,1} = \tfrac{1}{2} \tag{3.100b}$$

$$A_2 = -\tfrac{1}{6} \tag{3.100c}$$

$$A_{2,2} = \tfrac{1}{12} \tag{3.100d}$$

so that the bass boost factor is

$$\begin{aligned}
B &= \left| \frac{1}{6} + \frac{1}{2}\cos(q)\cos(f)\left(\frac{1+jkr}{jkr}\right) + \left(-\frac{1}{6}\frac{1}{2}[3\sin^2(f)-1] + \frac{1}{12}3\cos(2q)\cos^2(f)\right) \right. \\
&\qquad \left. \times\left(\frac{-3-j3kr+k^2r^2}{k^2r^2}\right) \right| \\
&= \frac{1}{2}\left| \frac{1}{3} + \cos(q)\cos(f)\left(\frac{1+jkr}{jkr}\right) + \left(-\frac{1}{6}\left[3\sin^2(f)-1\right] + \frac{1}{2}\cos(2q)\cos^2(f)\right) \right. \\
&\qquad \left. \times\left(\frac{-3-j3kr+k^2r^2}{k^2r^2}\right) \right| \\
&= \frac{1}{2}\left| \frac{1}{3} + \cos(q)\cos(f)\left(\frac{1+jkr}{jkr}\right) + \left(-\frac{1}{2}\sin^2(f) + \frac{1}{6} + \frac{1}{2}\cos(2q)\cos^2(f)\right) \right. \\
&\qquad \left. \times\left(\frac{-3-j3kr+k^2r^2}{k^2r^2}\right) \right| \\
&= \frac{1}{2}\left| \frac{1}{3} + \cos(q)\cos(f)\left(\frac{1+jkr}{jkr}\right) + \left(-\frac{1}{2}\sin^2(f) + \frac{1}{6} + \frac{1}{2}\cos^2(q)\cos^2(f) \right.\right. \\
&\qquad \left.\left. -\frac{1}{2}\sin^2(q)\cos^2(f)\right)\left(\frac{-3-j3kr+k^2r^2}{k^2r^2}\right) \right| \\
&= \frac{1}{2}\left| \frac{1}{3} + \cos(q)\cos(f)\left(\frac{1+jkr}{jkr}\right) + \left(-\frac{1}{2}\sin^2(f) + \frac{1}{6} + \frac{1}{2}\cos^2(q)\cos^2(f) \right.\right. \\
&\qquad \left.\left. -\frac{1}{2}\cos^2(f) + \frac{1}{2}\cos^2(q)\cos^2(f)\right)\left(\frac{-3-j3kr+k^2r^2}{k^2r^2}\right) \right| \\
&= \frac{1}{2}\left| \frac{1}{3} + \cos(q)\cos(f)\left(\frac{1+jkr}{jkr}\right) + \left(\frac{1}{6} - \frac{1}{2} + \cos^2(q)\cos^2(f)\right) \right. \\
&\qquad \left. \times\left(\frac{-3-j3kr+k^2r^2}{k^2r^2}\right) \right| \\
&= \frac{1}{2}\left| \frac{1}{3} - \frac{-3-j3kr+k^2r^2}{3k^2r^2} + \cos(q)\cos(f)\left(\frac{1+jkr}{jkr}\right) \right. \\
&\qquad \left. + \cos^2(q)\cos^2(f)\left(\frac{-3-j3kr+k^2r^2}{k^2r^2}\right) \right| \\
&= \frac{1}{2}\left| \frac{1+jkr}{k^2r^2} + \cos(q)\cos(f)\left(\frac{1+jkr}{jkr}\right) + \cos^2(q)\cos^2(f)\left(\frac{-3-j3kr+k^2r^2}{k^2r^2}\right) \right|
\end{aligned} \tag{3.101}$$

Graphs of the bass boost for frontal, 90° off-axis, and rear incidence are given for first-order and second-order cardioid microphones in figure 3.9.

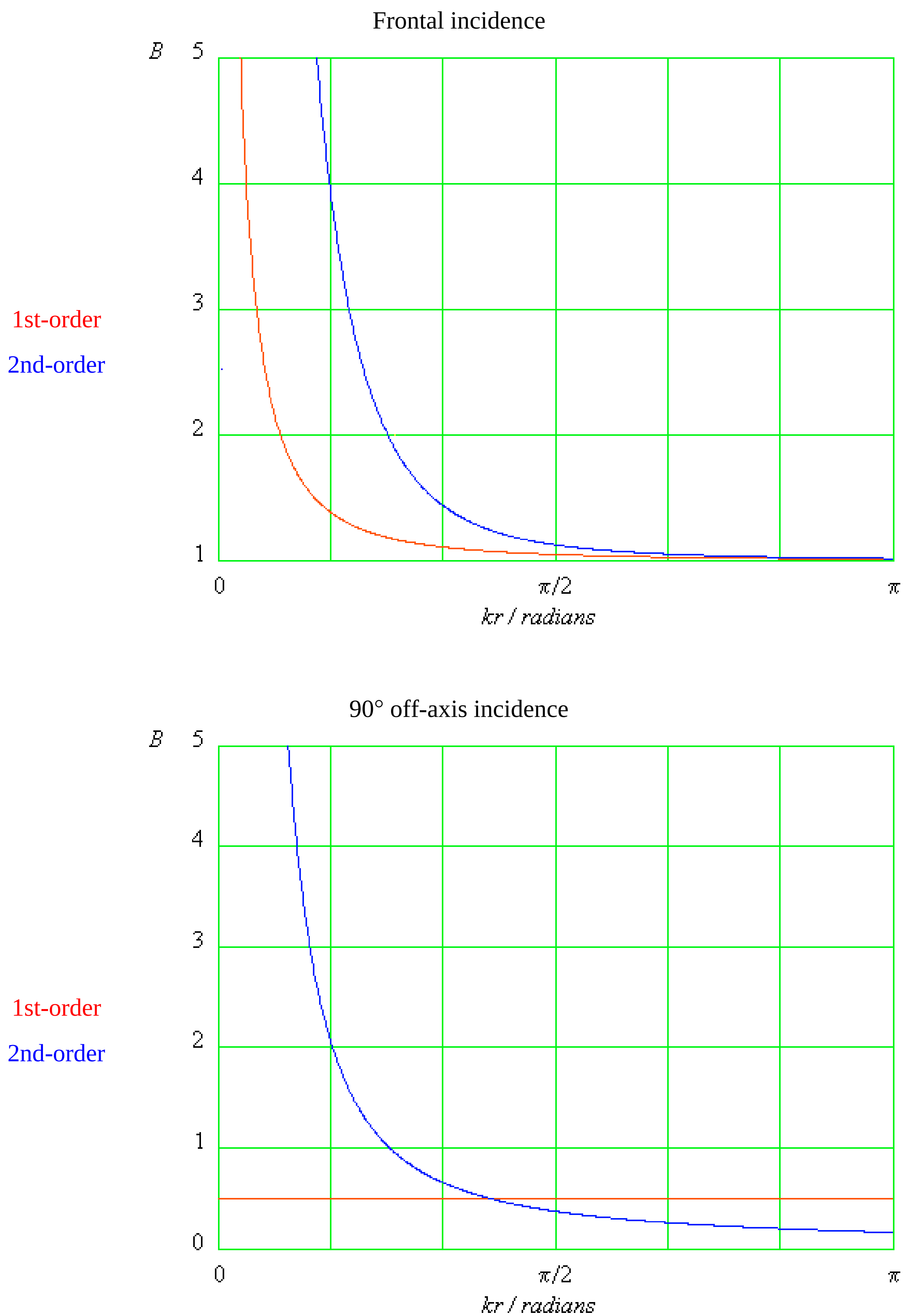


Figure 3.9: Bass Boost for First-Order and Second-Order Cardioid Microphones

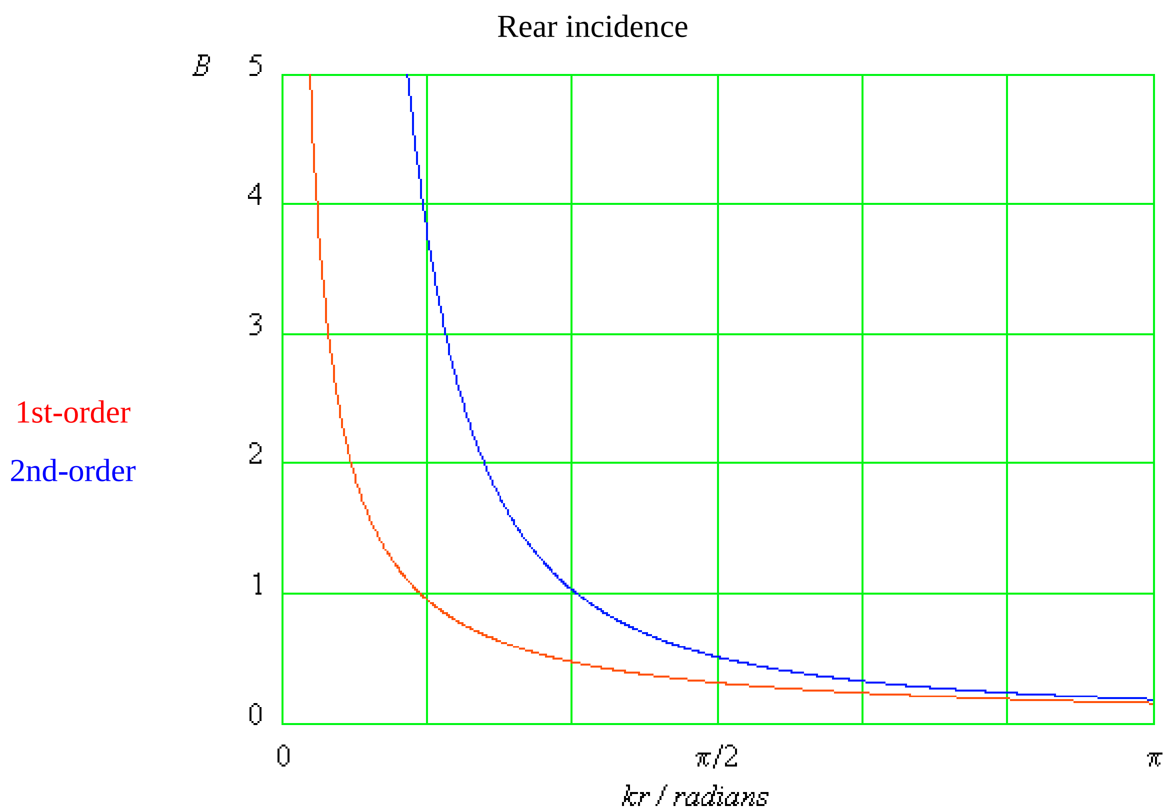


Figure 3.9: Bass Boost for First-Order and Second-Order Cardioid Microphones (continued)

3.6: The Blumlein Difference Technique

The "Blumlein difference technique", so named by Gerzon [37] [51], is a method of using microphones of a given order to obtain directional responses of higher order. The second-order soundfield microphone depends on this technique to generate second-order responses using first-order capsules.

We consider first the use of two pressure microphones to derive a first-order pressure gradient response. The directional derivative of the sound pressure field may be approximated by the difference in pressure between two points separated by a small distance $2d$;

$$\hat{\mathbf{u}}_1 \cdot \nabla p \approx \frac{p(\mathbf{x} + d\,\hat{\mathbf{u}}_1) - p(\mathbf{x} - d\,\hat{\mathbf{u}}_1)}{2d} \tag{3.102}$$

Physically, this may be realised by placing two (omnidirectional) pressure microphones a short distance apart, and taking the difference of their outputs. This method was used by Blumlein to obtain a dipole response before the availability of individual pressure gradient capsules [63]. Combined pressure / pressure gradient responses may be obtained by forming a suitably weighted sum of the equalised difference signal and the direct output of one of the individual capsules or, for better results, of a third capsule located midway between them [37] [38] [70].

Since

$$\hat{\mathbf{u}}_1 \cdot \nabla(\hat{\mathbf{u}}_1 \cdot \nabla p) \approx \frac{\hat{\mathbf{u}}_1 \cdot \nabla p\big|_{\mathbf{x}+d\,\hat{\mathbf{u}}_1} - \hat{\mathbf{u}}_1 \cdot \nabla p\big|_{\mathbf{x}-d\,\hat{\mathbf{u}}_1}}{2d} \qquad (3.103)$$

an axial quadrupole may be obtained by taking the difference between the outputs of two first-order pressure gradient microphones positioned such that each has the same directivity axis $\hat{\mathbf{u}}_1$ and the relative position of one with respect to the other is $2d\,\hat{\mathbf{u}}_1$; see figure 3.10.

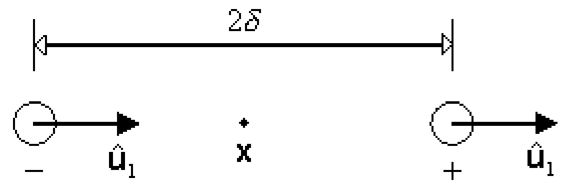


Figure 3.10: Axial Quadrupole as Difference of Two First-Order Pressure Gradient Microphones

This was described by Olson (who also noted that the technique may be extended to higher orders, since an *n*th-order pressure gradient may be obtained by taking the difference of two gradients of order $n-1$) [70] [74]. It will be appreciated that the same result may be obtained by taking the sum of two coaxially positioned dipoles pointing in opposite directions.

If general first-order microphones are used instead of dipoles, we obtain (omitting for clarity the $G/(a+b)$ scale factor)

$$\begin{aligned}
&\frac{\left(ap + bc\int(\hat{\mathbf{u}}_1 \cdot \nabla p)dt\right)\Big|_{\mathbf{x}+d\,\hat{\mathbf{u}}_1} - \left(ap + bc\int(\hat{\mathbf{u}}_1 \cdot \nabla p)dt\right)\Big|_{\mathbf{x}-d\,\hat{\mathbf{u}}_1}}{2d} \\
&\qquad = \frac{a}{2d}\left(p\big|_{\mathbf{x}+d\,\hat{\mathbf{u}}_1} - p\big|_{\mathbf{x}-d\,\hat{\mathbf{u}}_1}\right) + \frac{bc}{2d}\int\left(\hat{\mathbf{u}}_1 \cdot \nabla p\big|_{\mathbf{x}+d\,\hat{\mathbf{u}}_1} - \hat{\mathbf{u}}_1 \cdot \nabla p\big|_{\mathbf{x}-d\,\hat{\mathbf{u}}_1}\right)dt \\
&\qquad \approx a\hat{\mathbf{u}}_1 \cdot \nabla p + bc\int\left(\hat{\mathbf{u}}_1 \cdot \nabla(\hat{\mathbf{u}}_1 \cdot \nabla p)\right)dt
\end{aligned} \tag{3.104}$$

Here the difference of pressure components yields a first-order gradient term, while the difference of first-order pressure gradients contributes a second-order gradient element. In the particular case $a = b,$ a second-order cardioid is obtained; this was also described by Olson [70] [74] [81]. Note that care is needed with such arrangements to ensure that appropriate equalisation is applied to signals representing different orders of gradient.

We may make a more general statement than that expressed by equation (3.103) as follows:

$$\hat{\mathbf{u}}_2 \cdot \nabla(\hat{\mathbf{u}}_1 \cdot \nabla p) \approx \frac{\hat{\mathbf{u}}_1 \cdot \nabla p\big|_{\mathbf{x}+d\,\hat{\mathbf{u}}_2} - \hat{\mathbf{u}}_1 \cdot \nabla p\big|_{\mathbf{x}-d\,\hat{\mathbf{u}}_2}}{2d} \tag{3.105}$$

where the two vectors may be chosen independently. A tesseral quadrupole may therefore be obtained by positioning two dipoles such that their directivity axes are parallel, so that $\hat{\mathbf{u}}_2$ and $\hat{\mathbf{u}}_1$ are perpendicular; Gerzon described this approach [37].

We may generalise further by considering the difference between the outputs of two first-order microphones of arbitrary polar response. A very large number of second-order polar responses may then be obtained; the possibilities may be further extended by combining the equalised difference signal with the output from a third microphone located between the Blumlein difference pair. Some examples are discussed in [37] [38]. It will be appreciated that, in principle, the polar response and orientation of each of the microphones could be chosen independently; this will not be pursued further here, since the principles are adequately demonstrated by the simpler cases already considered.

A satisfactory Blumlein difference signal can be obtained from a pair of microphones with any given separation only over a restricted frequency range. At frequencies where the distance between the microphones is no longer sufficiently small compared to the wavelength, the desired polar response is not obtained. Gerzon stated that the technique performs “reasonably well” up to frequencies where the microphone separation is equal to half the wavelength, although such a judgement is necessarily somewhat inexact, since it depends on the amount of variation from the nominal response which is considered acceptable. At low frequencies, the difference in sound pressure between the microphones becomes small, and the Blumlein difference signal therefore becomes small compared to spurious signal components arising from difference between the microphones. These limitations are discussed in detail in [38] and [39].

Each of the first-order pressure gradients in equation (3.103) could be derived by using a pair of pressure microphones rather than a dipole; this would represent a system employing a linear array of four pressure microphones to obtain the second-order pressure gradient. Second-order microphones based on this principle have been investigated [78] [79] [82], but we will not consider them here since this approach is not directly relevant to the second-order soundfield microphone. Other techniques which have been used to obtain second-order microphones include the use of microphone capsules embedded in the side of a hollow cylinder [80], and the placing of a single dipole close to a plane surface to exploit the boundary effect, effectively taking the difference between the output of the real microphone and its acoustic image [25].

We will now consider the use of a rectangular array of four first-order capsules to derive a tesseral quadrupole in a manner which may be considered to be an extension of the Blumlein difference technique. It will be seen in Chapter 6 that arrangements of this sort occur in the second-order soundfield microphone.

Let the four capsules be denoted FL, BL, FR and BR. The positions of the capsules are

$$\mathbf{x}_{FL} = \left[\tfrac{1}{2}d_2 \quad \tfrac{1}{2}d_1 \quad 0\right]^T \tag{3.106a}$$
$$\mathbf{x}_{BL} = \left[-\tfrac{1}{2}d_2 \quad \tfrac{1}{2}d_1 \quad 0\right]^T \tag{3.106b}$$
$$\mathbf{x}_{FR} = \left[\tfrac{1}{2}d_2 \quad -\tfrac{1}{2}d_1 \quad 0\right]^T \tag{3.106c}$$
$$\mathbf{x}_{BR} = \left[-\tfrac{1}{2}d_2 \quad -\tfrac{1}{2}d_1 \quad 0\right]^T \tag{3.106d}$$

and their directivity axes are

$$\hat{\mathbf{u}}_{FL} = \left[\cos(a) \quad \sin(a) \quad 0\right]^T \tag{3.107a}$$
$$\hat{\mathbf{u}}_{BL} = \left[-\cos(a) \quad \sin(a) \quad 0\right]^T \tag{3.107b}$$

$$\hat{\mathbf{u}}_{FR} = [\cos(a) \quad -\sin(a) \quad 0]^T \tag{3.107c}$$
$$\hat{\mathbf{u}}_{BR} = [-\cos(a) \quad -\sin(a) \quad 0]^T \tag{3.107d}$$

This arrangement is shown in figure 3.11.

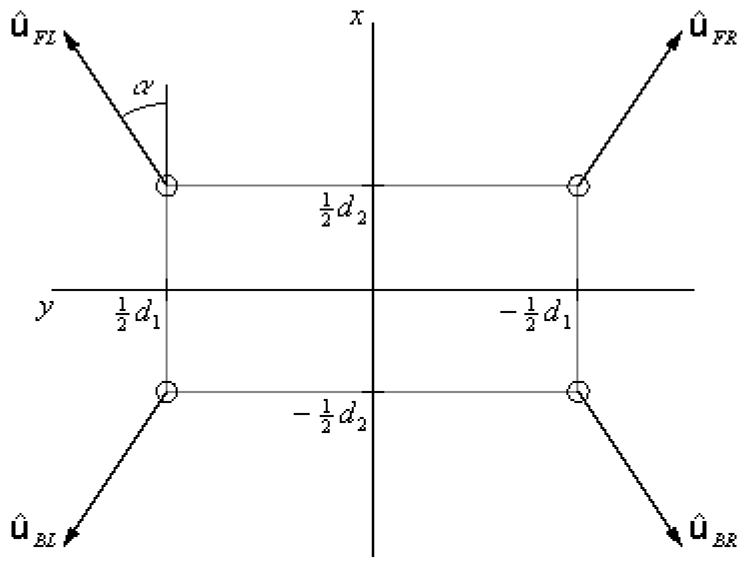


Figure 3.11: Capsule Configuration for Derivation of Tesseral Quadrupole Response

Let the sound pressure at the centre of the array be

$$p_O = Ae^{jwt} \tag{3.108}$$

The sound pressure at each of the capsules is then given by

$$p_{FL} = e^{jk\hat{\mathbf{d}}\cdot\mathbf{x}_{FL}}\, p_O \tag{3.109a}$$
$$p_{BL} = e^{jk\hat{\mathbf{d}}\cdot\mathbf{x}_{BL}}\, p_O \tag{3.109b}$$
$$p_{FR} = e^{jk\hat{\mathbf{d}}\cdot\mathbf{x}_{FR}}\, p_O \tag{3.109c}$$
$$p_{BR} = e^{jk\hat{\mathbf{d}}\cdot\mathbf{x}_{BR}}\, p_O \tag{3.109d}$$

and the capsule output signals are

$$v_{FL} = \frac{G}{a+b}\left[a + b\hat{\mathbf{u}}_{FL} \cdot \hat{\mathbf{d}}\right] e^{jk\hat{\mathbf{d}}\cdot\mathbf{x}_{FL}}\, p_O \tag{3.110a}$$

$$v_{BL} = \frac{G}{a+b}\left[a + b\hat{\mathbf{u}}_{BL} \cdot \hat{\mathbf{d}}\right] e^{jk\hat{\mathbf{d}}\cdot\mathbf{x}_{BL}} p_O \tag{3.110b}$$

$$v_{FR} = \frac{G}{a+b}\left[a + b\hat{\mathbf{u}}_{FR} \cdot \hat{\mathbf{d}}\right] e^{jk\hat{\mathbf{d}}\cdot\mathbf{x}_{FR}} p_O \tag{3.110c}$$

$$v_{BR} = \frac{G}{a+b}\left[a + b\hat{\mathbf{u}}_{BR} \cdot \hat{\mathbf{d}}\right] e^{jk\hat{\mathbf{d}}\cdot\mathbf{x}_{BR}} p_O \tag{3.110d}$$

Let V_B be the difference signal

$$\begin{aligned} V_B &= (v_{FL} - v_{BL}) - (v_{FR} - v_{BR}) \\ &= v_{FL} - v_{BL} - v_{FR} + v_{BR} \end{aligned} \tag{3.111}$$

Substituting in the capsule outputs

$$\begin{aligned} V_B &= \frac{G}{a+b}\Big\{\left[a + b\hat{\mathbf{u}}_{FL} \cdot \hat{\mathbf{d}}\right] e^{jk\hat{\mathbf{d}}\cdot\mathbf{x}_{FL}} - \left[a + b\hat{\mathbf{u}}_{BL} \cdot \hat{\mathbf{d}}\right] e^{jk\hat{\mathbf{d}}\cdot\mathbf{x}_{BL}} \\ &\qquad - \left[a + b\hat{\mathbf{u}}_{FR} \cdot \hat{\mathbf{d}}\right] e^{jk\hat{\mathbf{d}}\cdot\mathbf{x}_{FR}} + \left[a + b\hat{\mathbf{u}}_{BR} \cdot \hat{\mathbf{d}}\right] e^{jk\hat{\mathbf{d}}\cdot\mathbf{x}_{BR}} \Big\} p_O \\ &= \frac{G}{a+b}\Big\{a\left[e^{jk\hat{\mathbf{d}}\cdot\mathbf{x}_{FL}} - e^{jk\hat{\mathbf{d}}\cdot\mathbf{x}_{BL}} - e^{jk\hat{\mathbf{d}}\cdot\mathbf{x}_{FR}} + e^{jk\hat{\mathbf{d}}\cdot\mathbf{x}_{BR}}\right] \\ &\qquad + b\left[\hat{\mathbf{u}}_{FL} e^{jk\hat{\mathbf{d}}\cdot\mathbf{x}_{FL}} - \hat{\mathbf{u}}_{BL} e^{jk\hat{\mathbf{d}}\cdot\mathbf{x}_{BL}} - \hat{\mathbf{u}}_{FR} e^{jk\hat{\mathbf{d}}\cdot\mathbf{x}_{FR}} + \hat{\mathbf{u}}_{BR} e^{jk\hat{\mathbf{d}}\cdot\mathbf{x}_{BR}}\right] \cdot \hat{\mathbf{d}}\Big\} p_O \end{aligned} \tag{3.112}$$

Now since $\mathbf{x}_{BR} = -\mathbf{x}_{FL}$, $\mathbf{x}_{FR} = -\mathbf{x}_{BL}$, $\hat{\mathbf{u}}_{BR} = -\hat{\mathbf{u}}_{FL}$ and $\hat{\mathbf{u}}_{FR} = -\hat{\mathbf{u}}_{BL}$, so

$$\begin{aligned} V_B &= \frac{G}{a+b}\Big\{a\left[e^{jk\hat{\mathbf{d}}\cdot\mathbf{x}_{FL}} - e^{jk\hat{\mathbf{d}}\cdot\mathbf{x}_{BL}} - e^{-jk\hat{\mathbf{d}}\cdot\mathbf{x}_{BL}} + e^{-jk\hat{\mathbf{d}}\cdot\mathbf{x}_{FL}}\right] \\ &\qquad + b\left[\hat{\mathbf{u}}_{FL} e^{jk\hat{\mathbf{d}}\cdot\mathbf{x}_{FL}} - \hat{\mathbf{u}}_{BL} e^{jk\hat{\mathbf{d}}\cdot\mathbf{x}_{BL}} + \hat{\mathbf{u}}_{BL} e^{-jk\hat{\mathbf{d}}\cdot\mathbf{x}_{BL}} - \hat{\mathbf{u}}_{FL} e^{-jk\hat{\mathbf{d}}\cdot\mathbf{x}_{FL}}\right] \cdot \hat{\mathbf{d}}\Big\} p_0 \\ &= \frac{G}{a+b}\Big\{a\left[e^{jk\hat{\mathbf{d}}\cdot\mathbf{x}_{FL}} + e^{-jk\hat{\mathbf{d}}\cdot\mathbf{x}_{FL}} - e^{jk\hat{\mathbf{d}}\cdot\mathbf{x}_{BL}} - e^{-jk\hat{\mathbf{d}}\cdot\mathbf{x}_{BL}}\right] \\ &\qquad + b\left[\hat{\mathbf{u}}_{FL} e^{jk\hat{\mathbf{d}}\cdot\mathbf{x}_{FL}} - \hat{\mathbf{u}}_{FL} e^{-jk\hat{\mathbf{d}}\cdot\mathbf{x}_{FL}} - \hat{\mathbf{u}}_{BL} e^{jk\hat{\mathbf{d}}\cdot\mathbf{x}_{BL}} + \hat{\mathbf{u}}_{BL} e^{-jk\hat{\mathbf{d}}\cdot\mathbf{x}_{BL}}\right] \cdot \hat{\mathbf{d}}\Big\} p_0 \\ &= \frac{G}{a+b}\Big\{2a\left[\cos(k\hat{\mathbf{d}}\cdot\mathbf{x}_{FL}) - \cos(k\hat{\mathbf{d}}\cdot\mathbf{x}_{BL})\right] \\ &\qquad + j2b\left[\hat{\mathbf{u}}_{FL}\sin(k\hat{\mathbf{d}}\cdot\mathbf{x}_{FL}) - \hat{\mathbf{u}}_{BL}\sin(k\hat{\mathbf{d}}\cdot\mathbf{x}_{BL})\right] \cdot \hat{\mathbf{d}}\Big\} p_O \end{aligned} \tag{3.113}$$

If the capsule separations are sufficiently small compared to the wavelength, then the scalar products $k\hat{\mathbf{d}}\cdot\mathbf{x}_{FL}$ and $k\hat{\mathbf{d}}\cdot\mathbf{x}_{FL}$ are small compared to unity, and we may write

$$
\begin{aligned}
V_B &= \frac{G}{a+b}\left\{2a[1-1] + j2b\left[\hat{\mathbf{u}}_{FL}(k\hat{\mathbf{d}}\cdot\mathbf{x}_{FL}) - \hat{\mathbf{u}}_{BL}(k\hat{\mathbf{d}}\cdot\mathbf{x}_{BL})\right]\cdot\hat{\mathbf{d}}\right\}p_O \\
&= \frac{j2Gb}{a+b}\left(\begin{bmatrix}\cos(a)\\ \sin(a)\\ 0\end{bmatrix}\tfrac{1}{2}k\left(d_2\cos(q)\cos(f) + d_1\sin(q)\cos(f)\right)\right. \\
&\qquad \left. - \begin{bmatrix}-\cos(a)\\ \sin(a)\\ 0\end{bmatrix}\tfrac{1}{2}k\left(-d_2\cos(q)\cos(f) + d_1\sin(q)\cos(f)\right)\right)\cdot\hat{\mathbf{d}}p_O \\
&= \frac{jGbk}{a+b}\left(\begin{bmatrix}d_2\cos(a)\cos(q)\cos(f) + d_1\cos(a)\sin(q)\cos(f)\\ d_2\sin(a)\cos(q)\cos(f) + d_1\sin(a)\sin(q)\cos(f)\\ 0\end{bmatrix}\right. \\
&\qquad \left. - \begin{bmatrix}d_2\cos(a)\cos(q)\cos(f) - d_1\cos(a)\sin(q)\cos(f)\\ -d_2\sin(a)\cos(q)\cos(f) + d_1\sin(a)\sin(q)\cos(f)\\ 0\end{bmatrix}\right)\cdot\hat{\mathbf{d}}p_O \\
&= \frac{jGbk}{a+b}\begin{bmatrix}2d_1\cos(a)\sin(q)\cos(f)\\ 2d_2\sin(a)\cos(q)\cos(f)\\ 0\end{bmatrix}\cdot\begin{bmatrix}\cos(q)\cos(f)\\ \sin(q)\cos(f)\\ \sin(f)\end{bmatrix}p_O \\
&= \frac{jGbk}{a+b}2\left(d_1\cos(a) + d_2\sin(a)\right)\sin(q)\cos(q)\cos^2(f)\,p_O \\
&= jk\frac{Gb}{a+b}\left(d_1\cos(a) + d_2\sin(a)\right)\sin(2q)\cos^2(f)\,p_O
\end{aligned}
\qquad (3.114)
$$

which is a tesseral quadrupole response.