6: The Second-Order Soundfield Microphone

Gerzon stated [40] that a second-order soundfield microphone may be constructed "...using twelve small cardioid or hypercardioid capsules mounted to form the faces of a regular dodecahedron having a small diameter ... the second-harmonic aspects of the directional pickup can be derived from these by techniques similar to the Blumlein difference technique." Since the dodecahedron is the simplest regular polyhedron with nine or more faces, so a dodecahedral array is the simplest arrangement of microphones from which the nine independent signals comprising the second-order B-format set can be obtained.

6.1: Geometry of the Dodecahedron

It is desirable to maximise the degree of symmetry present in the coefficients of the A-B matrix, since this simplifies the mathematical treatment (and will also simplify the eventual implementation).

The symmetry which is apparent in the coefficients of the A-B matrix in the case of the first-order soundfield microphone is related to the symmetry of the tetrahedron; specifically, to the presence of C_3 operations in the group of rotational symmetries of the tetrahedron [61]. Each C_3 axis passes through a vertex and the centroid of the opposite face of the tetrahedron. Rotation by 120° about one of these axes has the effect of taking $\cos(q)\cos(f)$ to $\sin(q)\cos(f)$ (or vice versa) and etc.; i.e., of inducing a cyclic permutation of the polar patterns associated with the first-order spherical harmonic component signals. The same rotation interchanges the faces of the tetrahedron in a corresponding manner.

Such C_3 operations are also present in the group of symmetries of the dodecahedron. To maximise the symmetry in the A-B matrix coefficients for the second-order soundfield microphone, the dodecahedral capsule array is oriented such that appropriate C_3 axes (which pass through two diametrically opposed vertices) are aligned with those of the tetrahedral first-order soundfield microphone array.

By inspection, two orientations which satisfy this criterion may be identified; selection between these is arbitrary. The chosen orientation is such that the highest part of the dodecahedron is an edge running front-back, and the front-most part is a horizontal edge - see figure 6.1.

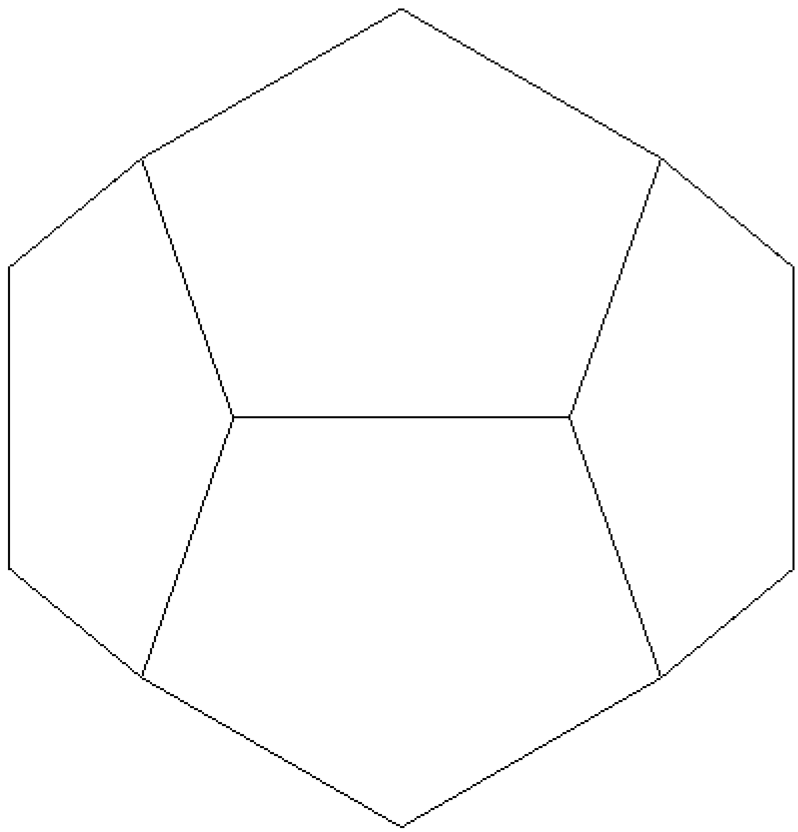

Figure 6.1: Orientation of Dodecahedron
(viewed from centre-front direction)

Two faces, having the front-most edge in common, point symmetrically up and down, with no left / right component in their orientation; these faces are conveniently labelled *FU* and *FD*. Their (outward) unit surface normal vectors are

$$\hat{\mathbf{u}}_{FU} = \begin{bmatrix} x \\ 0 \\ z \end{bmatrix} \tag{6.1}$$

and

$$\hat{\mathbf{u}}_{FD} = \begin{bmatrix} x \\ 0 \\ -z \end{bmatrix} \tag{6.2}$$

The cosine of the angle between vectors normal to two adjacent faces of a regular polyhedron is equal to the cosine of the dihedral angel at each edge of that polyhedron; hence, the scalar product of $\hat{\mathbf{u}}_{FU}$ and $\hat{\mathbf{u}}_{FD}$ is equal to the cosine of the dihedral angle at the edges of a regular dodecahedron.

Now, for any regular polyhedron,

$$J_d = 2\sin^{-1}\left(\frac{\cos(p/e_v)}{\sin(p/e_f)}\right) \tag{6.3}$$

where J_d is the dihedral angle, e_f is the number of edges around each face, and e_v is the number of edges which meet at each vertex [66]. Rearranging gives

$$\sin\left(\frac{J_d}{2}\right) = \frac{\cos(p/e_v)}{\sin(p/e_f)} \tag{6.4}$$

For a dodecahedron, $e_v = 3$ and $e_f = 5;$ hence

$$\begin{aligned} \sin\left(\frac{J_d}{2}\right) &= \frac{\cos(p/3)}{\sin(p/5)} \\ &= \frac{\frac{1}{2}}{\frac{1}{4}\sqrt{2}\sqrt{5-\sqrt{5}}} \\ &= \frac{\sqrt{2}}{\sqrt{5-\sqrt{5}}} \end{aligned} \tag{6.5}$$

Now since, for any angle j,

$$\cos(j) = \pm\sqrt{1-\sin^2(j)} \tag{6.6}$$

so we may use the trigonometric “double angle” identity for sines to obtain

$$
\begin{aligned}
\sin(J_d) &= 2\sin\left(\frac{J_d}{2}\right)\cos\left(\frac{J_d}{2}\right) \\
&= 2\sin\left(\frac{J_d}{2}\right)\sqrt{1-\sin^2\left(\frac{J_d}{2}\right)} \\
&= 2\frac{\sqrt{2}}{\sqrt{5-\sqrt{5}}}\sqrt{1-\frac{2}{5-\sqrt{5}}} \\
&= \frac{2\sqrt{2}}{\sqrt{5-\sqrt{5}}}\sqrt{\frac{3-\sqrt{5}}{5-\sqrt{5}}} \\
&= \frac{2\sqrt{2}\sqrt{3-\sqrt{5}}}{5-\sqrt{5}} \\
&= \frac{\sqrt{3-\sqrt{5}}\left(\sqrt{5}+1\right)}{\sqrt{10}}
\end{aligned}
\qquad (6.7)
$$

where the positive square root is taken in the substitution from equation (6.6) because the dihedral angle must by definition be less than 180°. We can now find the cosine of the dihedral angle:

$$
\begin{aligned}
\cos(J_d) &= \sqrt{1-\sin^2(J_d)} \\
&= \sqrt{1-\left(\frac{\sqrt{3-\sqrt{5}}\left(1+\sqrt{5}\right)}{\sqrt{10}}\right)^2} \\
&= \sqrt{1-\frac{\left(3-\sqrt{5}\right)\left(6+2\sqrt{5}\right)}{10}} \\
&= \sqrt{\frac{2}{10}} \\
&= \frac{1}{\sqrt{5}}
\end{aligned}
\qquad (6.8)
$$

where the positive square root is taken since (by inspection) the angle in question is less than 90°.

We can now determine the values of the elements of $\hat{\mathbf{u}}_{FU}$ and $\hat{\mathbf{u}}_{FD}$. Since they are unit vectors,

$$x^2 + z^2 = 1 \tag{6.9}$$

and since their scalar product is equal to $1/\sqrt{5}$,

$$x^2 - z^2 = \frac{1}{\sqrt{5}} \tag{6.10}$$

Equations (6.9) and (6.10) may be solved to give

$$\begin{aligned} x &= \sqrt{\frac{1}{2}\left(1+\frac{1}{\sqrt{5}}\right)} \\ &= \sqrt{\frac{\sqrt{5}}{2}\frac{\sqrt{5}+1}{5}} \\ &= \sqrt{\frac{1}{10}}\sqrt{5+\sqrt{5}} \end{aligned} \tag{6.11}$$

and

$$\begin{aligned} z &= \sqrt{\frac{1}{2}\left(1-\frac{1}{\sqrt{5}}\right)} \\ &= \sqrt{\frac{\sqrt{5}}{2}\frac{\sqrt{5}-1}{5}} \\ &= \sqrt{\frac{1}{10}}\sqrt{5-\sqrt{5}} \end{aligned} \tag{6.12}$$

so that the vectors are

$$\hat{\mathbf{u}}_{FU} = \begin{bmatrix} \sqrt{\frac{1}{10}}\sqrt{5+\sqrt{5}} \\ 0 \\ \sqrt{\frac{1}{10}}\sqrt{5-\sqrt{5}} \end{bmatrix} \tag{6.13}$$

and

$$\hat{\mathbf{u}}_{FD} = \begin{bmatrix} \sqrt{\frac{1}{10}}\sqrt{5+\sqrt{5}} \\ 0 \\ -\sqrt{\frac{1}{10}}\sqrt{5-\sqrt{5}} \end{bmatrix} \tag{6.14}$$

The opposite (backward-facing) faces, labelled *BU* and *BD*, necessarily have unit surface normal vectors which are obtained by multiplying $\hat{\mathbf{u}}_{FD}$ and $\hat{\mathbf{u}}_{FU}$ by -1:

$$\hat{\mathbf{u}}_{BU} = \begin{bmatrix} -\sqrt{\frac{1}{10}}\sqrt{5+\sqrt{5}} \\ 0 \\ \sqrt{\frac{1}{10}}\sqrt{5-\sqrt{5}} \end{bmatrix} \tag{6.15}$$

and

$$\hat{\mathbf{u}}_{BD} = \begin{bmatrix} -\sqrt{\frac{1}{10}}\sqrt{5+\sqrt{5}} \\ 0 \\ -\sqrt{\frac{1}{10}}\sqrt{5-\sqrt{5}} \end{bmatrix} \tag{6.16}$$

The remaining eight faces can similarly be grouped into pairs for which the surface normal vectors are equal in one component, equal and opposite in another, and zero in the third; furthermore, the vectors associated with each pair differ from those associated with the opposite pair only by a factor of -1. Therefore, calculations equivalent to those above may be used to obtain these vectors.

It is convenient to define

$$c^{+} = \sqrt{\frac{1}{10}}\sqrt{5+\sqrt{5}} = \frac{1}{10}\sqrt{50+10\sqrt{5}} \tag{6.17a}$$

$$c^{-} = \sqrt{\frac{1}{10}}\sqrt{5-\sqrt{5}} = \frac{1}{10}\sqrt{50-10\sqrt{5}} \tag{6.17b}$$

The unit surface normal vectors for the faces of the dodecahedron may then be expressed in terms of these two values as

$$\hat{\mathbf{u}}_{FU} = \begin{bmatrix} c^+ & 0 & c^- \end{bmatrix}^T \tag{6.18a}$$

$$\hat{\mathbf{u}}_{FD} = \begin{bmatrix} c^+ & 0 & -c^- \end{bmatrix}^T \tag{6.18b}$$

$$\hat{\mathbf{u}}_{BU} = \begin{bmatrix} -c^+ & 0 & c^- \end{bmatrix}^T \tag{6.18c}$$

$$\hat{\mathbf{u}}_{BD} = \begin{bmatrix} -c^+ & 0 & -c^- \end{bmatrix}^T \tag{6.18d}$$

$$\hat{\mathbf{u}}_{LF} = \begin{bmatrix} c^- & c^+ & 0 \end{bmatrix}^T \tag{6.18e}$$

$$\hat{\mathbf{u}}_{LB} = \begin{bmatrix} -c^- & c^+ & 0 \end{bmatrix}^T \tag{6.18f}$$

$$\hat{\mathbf{u}}_{RF} = \begin{bmatrix} c^- & -c^+ & 0 \end{bmatrix}^T \tag{6.18g}$$

$$\hat{\mathbf{u}}_{RB} = \begin{bmatrix} -c^- & -c^+ & 0 \end{bmatrix}^T \tag{6.18h}$$

$$\hat{\mathbf{u}}_{UL} = \begin{bmatrix} 0 & c^- & c^+ \end{bmatrix}^T \tag{6.18i}$$

$$\hat{\mathbf{u}}_{UR} = \begin{bmatrix} 0 & -c^- & c^+ \end{bmatrix}^T \tag{6.18j}$$

$$\hat{\mathbf{u}}_{DL} = \begin{bmatrix} 0 & c^- & -c^+ \end{bmatrix}^T \tag{6.18k}$$

$$\hat{\mathbf{u}}_{DR} = \begin{bmatrix} 0 & -c^- & -c^+ \end{bmatrix}^T \tag{6.18l}$$

The constants c^+ and c^- satisfy the following relationships:

$$(c^+)^2 = \frac{1}{2} + \frac{\sqrt{5}}{10} \tag{6.19a}$$

$$(c^-)^2 = \frac{1}{2} - \frac{\sqrt{5}}{10} \tag{6.19b}$$

$$(c^+)^2 + (c^-)^2 = 1 \tag{6.19c}$$

$$(c^+)^2 - (c^-)^2 = \frac{1}{\sqrt{5}} \tag{6.19d}$$

$$c^+ c^- = \frac{1}{\sqrt{5}} \tag{6.19e}$$

$$\frac{c^+}{c^-} = \frac{\sqrt{5}+1}{2} \tag{6.19f}$$

$$\frac{c^-}{c^+} = \frac{\sqrt{5}-1}{2} \tag{6.19g}$$

6.2: Derivation of the A-B Matrix

The method described in Chapter 5 for the derivation of the A-B matrix coefficients in the case of the first-order soundfield microphone is not applicable when the second-order soundfield microphone is considered. While we can write equations similar to equations (5.25) and (5.26) for any of the zeroth-order or first-order signals, this leaves us in each case with twelve unknown matrix coefficients and only four equations.

A different approach is therefore required. Let a signal $\tilde{H}$ be a general linear combination of the twelve A-format signals:

$$\begin{aligned}\tilde{H} = {} & g_{FU} v_{FU} + g_{FD} v_{FD} + g_{BU} v_{BU} + g_{BD} v_{BD} + g_{LF} v_{LF} + g_{LB} v_{LB} \\ & + g_{RF} v_{RF} + g_{RB} v_{RB} + g_{UL} v_{UL} + g_{UR} v_{UR} + g_{DL} v_{DL} + g_{DR} v_{DR}\end{aligned} \tag{6.20}$$

or, by substituting for each of the A-format signals,

$$\begin{aligned}\tilde{H} = \frac{G}{a+b} \Big[& g_{FU} \left[a + b\hat{\mathbf{u}}_{FU} \cdot \hat{\mathbf{d}}\right] e^{jkr\hat{\mathbf{u}}_{FU}\cdot\hat{\mathbf{d}}} + g_{FD} \left[a + b\hat{\mathbf{u}}_{FD} \cdot \hat{\mathbf{d}}\right] e^{jkr\hat{\mathbf{u}}_{FD}\cdot\hat{\mathbf{d}}} \\ & + g_{BU} \left[a + b\hat{\mathbf{u}}_{BU} \cdot \hat{\mathbf{d}}\right] e^{jkr\hat{\mathbf{u}}_{BU}\cdot\hat{\mathbf{d}}} + g_{BD} \left[a + b\hat{\mathbf{u}}_{BD} \cdot \hat{\mathbf{d}}\right] e^{jkr\hat{\mathbf{u}}_{BD}\cdot\hat{\mathbf{d}}} \\ & + g_{LF} \left[a + b\hat{\mathbf{u}}_{LF} \cdot \hat{\mathbf{d}}\right] e^{jkr\hat{\mathbf{u}}_{LF}\cdot\hat{\mathbf{d}}} + g_{LB} \left[a + b\hat{\mathbf{u}}_{LB} \cdot \hat{\mathbf{d}}\right] e^{jkr\hat{\mathbf{u}}_{LB}\cdot\hat{\mathbf{d}}} \\ & + g_{RF} \left[a + b\hat{\mathbf{u}}_{RF} \cdot \hat{\mathbf{d}}\right] e^{jkr\hat{\mathbf{u}}_{RF}\cdot\hat{\mathbf{d}}} + g_{RB} \left[a + b\hat{\mathbf{u}}_{RB} \cdot \hat{\mathbf{d}}\right] e^{jkr\hat{\mathbf{u}}_{RB}\cdot\hat{\mathbf{d}}} \\ & + g_{UL} \left[a + b\hat{\mathbf{u}}_{UL} \cdot \hat{\mathbf{d}}\right] e^{jkr\hat{\mathbf{u}}_{UL}\cdot\hat{\mathbf{d}}} + g_{UR} \left[a + b\hat{\mathbf{u}}_{UR} \cdot \hat{\mathbf{d}}\right] e^{jkr\hat{\mathbf{u}}_{UR}\cdot\hat{\mathbf{d}}} \\ & + g_{DL} \left[a + b\hat{\mathbf{u}}_{DL} \cdot \hat{\mathbf{d}}\right] e^{jkr\hat{\mathbf{u}}_{DL}\cdot\hat{\mathbf{d}}} + g_{DR} \left[a + b\hat{\mathbf{u}}_{DR} \cdot \hat{\mathbf{d}}\right] e^{jkr\hat{\mathbf{u}}_{DR}\cdot\hat{\mathbf{d}}} \Big] p_O\end{aligned} \tag{6.21}$$

We may obtain expressions in terms of the matrix coefficients g_{FU}, g_{FD}, etc., for the coefficient of each spherical harmonic component in the Laplace series expansion of $\tilde{H}$ by evaluating the integrals given in equation (2.10). For each B-format signal, we can then equate these expressions to the desired values of the coefficients; the resulting equations may then be solved to find the matrix coefficients.

Note that this method is applicable to the second-order component signals; a method employing a coincident capsule approximation could not be used, since it is the phase differences between the capsules that allow these signals to be obtained. Furthermore, this approach automatically includes the dependence of the Laplace series coefficients on *kr*, so that it is not necessary to determine the frequency response functions separately from the matrix coefficients.

We consider first the zeroth-order component of $\tilde{H}$:

$$
\begin{aligned}
A_0 &= \frac{1}{4p}\int_0^{2p}\int_{-p/2}^{p/2}\tilde{H}\cos(f)\,df\,dq \\
&= K_0\int_0^{2p}\int_{-p/2}^{p/2}\Big\{g_{FU}\left[a+b\hat{\mathbf{u}}_{FU}\cdot\hat{\mathbf{d}}\right]e^{jkr\hat{\mathbf{u}}_{FU}\cdot\hat{\mathbf{d}}}+g_{FD}\left[a+b\hat{\mathbf{u}}_{FD}\cdot\hat{\mathbf{d}}\right]e^{jkr\hat{\mathbf{u}}_{FD}\cdot\hat{\mathbf{d}}} \\
&\qquad +g_{BU}\left[a+b\hat{\mathbf{u}}_{BU}\cdot\hat{\mathbf{d}}\right]e^{jkr\hat{\mathbf{u}}_{BU}\cdot\hat{\mathbf{d}}}+g_{BD}\left[a+b\hat{\mathbf{u}}_{BD}\cdot\hat{\mathbf{d}}\right]e^{jkr\hat{\mathbf{u}}_{BD}\cdot\hat{\mathbf{d}}} \\
&\qquad +g_{LF}\left[a+b\hat{\mathbf{u}}_{LF}\cdot\hat{\mathbf{d}}\right]e^{jkr\hat{\mathbf{u}}_{LF}\cdot\hat{\mathbf{d}}}+g_{LB}\left[a+b\hat{\mathbf{u}}_{LB}\cdot\hat{\mathbf{d}}\right]e^{jkr\hat{\mathbf{u}}_{LB}\cdot\hat{\mathbf{d}}} \\
&\qquad +g_{RF}\left[a+b\hat{\mathbf{u}}_{RF}\cdot\hat{\mathbf{d}}\right]e^{jkr\hat{\mathbf{u}}_{RF}\cdot\hat{\mathbf{d}}}+g_{RB}\left[a+b\hat{\mathbf{u}}_{RB}\cdot\hat{\mathbf{d}}\right]e^{jkr\hat{\mathbf{u}}_{RB}\cdot\hat{\mathbf{d}}} \\
&\qquad +g_{UL}\left[a+b\hat{\mathbf{u}}_{UL}\cdot\hat{\mathbf{d}}\right]e^{jkr\hat{\mathbf{u}}_{UL}\cdot\hat{\mathbf{d}}}+g_{UR}\left[a+b\hat{\mathbf{u}}_{UR}\cdot\hat{\mathbf{d}}\right]e^{jkr\hat{\mathbf{u}}_{UR}\cdot\hat{\mathbf{d}}} \\
&\qquad +g_{DL}\left[a+b\hat{\mathbf{u}}_{DL}\cdot\hat{\mathbf{d}}\right]e^{jkr\hat{\mathbf{u}}_{DL}\cdot\hat{\mathbf{d}}}+g_{DR}\left[a+b\hat{\mathbf{u}}_{DR}\cdot\hat{\mathbf{d}}\right]e^{jkr\hat{\mathbf{u}}_{DR}\cdot\hat{\mathbf{d}}}\Big\}\cos(f)\,df\,dq
\end{aligned}
\tag{6.22}
$$

where

$$K_0 = \frac{1}{4p}\frac{G}{a+b} \tag{6.23}$$

For compactness, we introduce the notation

$$L\left(f(q,f)\right) = \int_0^{2p}\int_{-p/2}^{p/2} f(q,f)\cos(f)\,df\,dq \tag{6.24}$$

Multiplying out the bracketed factors in equation (6.22) and using the linearity property of integration, we obtain

$$\begin{aligned}A_0 = K_0 \Big\{ & ag_{FU} L\left(e^{jkr\hat{\mathbf{u}}_{FU}\cdot\hat{\mathbf{d}}}\right) + bg_{FU} L\left(\hat{\mathbf{u}}_{FU}\cdot\hat{\mathbf{d}}e^{jkr\hat{\mathbf{u}}_{FU}\cdot\hat{\mathbf{d}}}\right)\\
&+ag_{FD} L\left(e^{jkr\hat{\mathbf{u}}_{FD}\cdot\hat{\mathbf{d}}}\right) + bg_{FD} L\left(\hat{\mathbf{u}}_{FD}\cdot\hat{\mathbf{d}}e^{jkr\hat{\mathbf{u}}_{FD}\cdot\hat{\mathbf{d}}}\right)\\
&+ag_{BU} L\left(e^{jkr\hat{\mathbf{u}}_{BU}\cdot\hat{\mathbf{d}}}\right) + bg_{BU} L\left(\hat{\mathbf{u}}_{FD}\cdot\hat{\mathbf{d}}e^{jkr\hat{\mathbf{u}}_{BU}\cdot\hat{\mathbf{d}}}\right)\\
&+ag_{BD} L\left(e^{jkr\hat{\mathbf{u}}_{BD}\cdot\hat{\mathbf{d}}}\right) + bg_{BD} L\left(\hat{\mathbf{u}}_{BD}\cdot\hat{\mathbf{d}}e^{jkr\hat{\mathbf{u}}_{BD}\cdot\hat{\mathbf{d}}}\right)\\
&+ag_{LF} L\left(e^{jkr\hat{\mathbf{u}}_{LF}\cdot\hat{\mathbf{d}}}\right) + bg_{LF} L\left(\hat{\mathbf{u}}_{LF}\cdot\hat{\mathbf{d}}e^{jkr\hat{\mathbf{u}}_{LF}\cdot\hat{\mathbf{d}}}\right)\\
&+ag_{LB} L\left(e^{jkr\hat{\mathbf{u}}_{LB}\cdot\hat{\mathbf{d}}}\right) + bg_{LB} L\left(\hat{\mathbf{u}}_{LB}\cdot\hat{\mathbf{d}}e^{jkr\hat{\mathbf{u}}_{LB}\cdot\hat{\mathbf{d}}}\right)\\
&+ag_{RF} L\left(e^{jkr\hat{\mathbf{u}}_{RF}\cdot\hat{\mathbf{d}}}\right) + bg_{RF} L\left(\hat{\mathbf{u}}_{RF}\cdot\hat{\mathbf{d}}e^{jkr\hat{\mathbf{u}}_{RF}\cdot\hat{\mathbf{d}}}\right)\\
&+ag_{RB} L\left(e^{jkr\hat{\mathbf{u}}_{RB}\cdot\hat{\mathbf{d}}}\right) + bg_{RB} L\left(\hat{\mathbf{u}}_{RB}\cdot\hat{\mathbf{d}}e^{jkr\hat{\mathbf{u}}_{RB}\cdot\hat{\mathbf{d}}}\right)\\
&+ag_{UL} L\left(e^{jkr\hat{\mathbf{u}}_{UL}\cdot\hat{\mathbf{d}}}\right) + bg_{UL} L\left(\hat{\mathbf{u}}_{UL}\cdot\hat{\mathbf{d}}e^{jkr\hat{\mathbf{u}}_{UL}\cdot\hat{\mathbf{d}}}\right)\\
&+ag_{UR} L\left(e^{jkr\hat{\mathbf{u}}_{UR}\cdot\hat{\mathbf{d}}}\right) + bg_{UR} L\left(\hat{\mathbf{u}}_{UR}\cdot\hat{\mathbf{d}}e^{jkr\hat{\mathbf{u}}_{UR}\cdot\hat{\mathbf{d}}}\right)\\
&+ag_{DL} L\left(e^{jkr\hat{\mathbf{u}}_{DL}\cdot\hat{\mathbf{d}}}\right) + bg_{DL} L\left(\hat{\mathbf{u}}_{DL}\cdot\hat{\mathbf{d}}e^{jkr\hat{\mathbf{u}}_{DL}\cdot\hat{\mathbf{d}}}\right)\\
&+ag_{DR} L\left(e^{jkr\hat{\mathbf{u}}_{DR}\cdot\hat{\mathbf{d}}}\right) + bg_{DR} L\left(\hat{\mathbf{u}}_{DR}\cdot\hat{\mathbf{d}}e^{jkr\hat{\mathbf{u}}_{DR}\cdot\hat{\mathbf{d}}}\right)\Big\}\end{aligned} \tag{6.25}$$

Rearranging gives

$$\begin{aligned}A_0 = aK_0 \Big\{ & g_{FU} L\left(e^{jkr\hat{\mathbf{u}}_{FD}\cdot\hat{\mathbf{d}}}\right) + g_{FD} L\left(e^{jkr\hat{\mathbf{u}}_{FD}\cdot\hat{\mathbf{d}}}\right)\\
&+g_{BU} L\left(e^{jkr\hat{\mathbf{u}}_{BU}\cdot\hat{\mathbf{d}}}\right) + g_{BD} L\left(e^{jkr\hat{\mathbf{u}}_{BD}\cdot\hat{\mathbf{d}}}\right)\\
&+g_{LF} L\left(e^{jkr\hat{\mathbf{u}}_{LF}\cdot\hat{\mathbf{d}}}\right) + g_{LB} L\left(e^{jkr\hat{\mathbf{u}}_{LB}\cdot\hat{\mathbf{d}}}\right)\\
&+g_{RF} L\left(e^{jkr\hat{\mathbf{u}}_{RF}\cdot\hat{\mathbf{d}}}\right) + g_{RB} L\left(e^{jkr\hat{\mathbf{u}}_{RB}\cdot\hat{\mathbf{d}}}\right)\\
&+g_{UL} L\left(e^{jkr\hat{\mathbf{u}}_{UL}\cdot\hat{\mathbf{d}}}\right) + g_{UR} L\left(e^{jkr\hat{\mathbf{u}}_{UR}\cdot\hat{\mathbf{d}}}\right)\\
&+g_{DL} L\left(e^{jkr\hat{\mathbf{u}}_{DL}\cdot\hat{\mathbf{d}}}\right) + g_{DR} L\left(e^{jkr\hat{\mathbf{u}}_{DR}\cdot\hat{\mathbf{d}}}\right)\Big\}\\
+bK_0 \Big\{ & g_{FU} L\left(\hat{\mathbf{u}}_{FU}\cdot\hat{\mathbf{d}}e^{jkr\hat{\mathbf{u}}_{FU}\cdot\hat{\mathbf{d}}}\right) + g_{FD} L\left(\hat{\mathbf{u}}_{FD}\cdot\hat{\mathbf{d}}e^{jkr\hat{\mathbf{u}}_{FD}\cdot\hat{\mathbf{d}}}\right)\\
&+g_{BU} L\left(\hat{\mathbf{u}}_{BU}\cdot\hat{\mathbf{d}}e^{jkr\hat{\mathbf{u}}_{BU}\cdot\hat{\mathbf{d}}}\right) + g_{BD} L\left(\hat{\mathbf{u}}_{BD}\cdot\hat{\mathbf{d}}e^{jkr\hat{\mathbf{u}}_{BD}\cdot\hat{\mathbf{d}}}\right)\\
&+g_{LF} L\left(\hat{\mathbf{u}}_{LF}\cdot\hat{\mathbf{d}}e^{jkr\hat{\mathbf{u}}_{LF}\cdot\hat{\mathbf{d}}}\right) + g_{LB} L\left(\hat{\mathbf{u}}_{LB}\cdot\hat{\mathbf{d}}e^{jkr\hat{\mathbf{u}}_{LB}\cdot\hat{\mathbf{d}}}\right)\\
&+g_{RF} L\left(\hat{\mathbf{u}}_{RF}\cdot\hat{\mathbf{d}}e^{jkr\hat{\mathbf{u}}_{RF}\cdot\hat{\mathbf{d}}}\right) + g_{RB} L\left(\hat{\mathbf{u}}_{RB}\cdot\hat{\mathbf{d}}e^{jkr\hat{\mathbf{u}}_{RB}\cdot\hat{\mathbf{d}}}\right)\\
&+g_{UL} L\left(\hat{\mathbf{u}}_{UL}\cdot\hat{\mathbf{d}}e^{jkr\hat{\mathbf{u}}_{UL}\cdot\hat{\mathbf{d}}}\right) + g_{UR} L\left(\hat{\mathbf{u}}_{UR}\cdot\hat{\mathbf{d}}e^{jkr\hat{\mathbf{u}}_{UR}\cdot\hat{\mathbf{d}}}\right)\\
&+g_{DL} L\left(\hat{\mathbf{u}}_{DL}\cdot\hat{\mathbf{d}}e^{jkr\hat{\mathbf{u}}_{DL}\cdot\hat{\mathbf{d}}}\right) + g_{DR} L\left(\hat{\mathbf{u}}_{DR}\cdot\hat{\mathbf{d}}e^{jkr\hat{\mathbf{u}}_{DR}\cdot\hat{\mathbf{d}}}\right)\Big\}\end{aligned} \tag{6.26}$$

It is now necessary to evaluate each of the integrals in this expression. We first observe that

$$\frac{d}{df'}\left\{-\frac{j}{kr}e^{jkr\sin(f')}\right\} = -\frac{j}{kr}\times jkre^{jkr\sin(f')}\times\frac{d\sin(f')}{df'} \qquad (6.27)$$
$$= e^{jkr\sin(f')}\cos(f')$$

so that (omitting the constant of integration)

$$\int e^{jkr\sin(f')}\cos(f')df' = -\frac{j}{kr}e^{jkr\sin(f')} \qquad (6.28)$$

Now consider the first integral in the above expression:

$$\int_0^{2p}\int_{-p/2}^{p/2} e^{jkr\hat{\mathbf{u}}_{FU}\cdot\hat{\mathbf{d}}}\cos(f)df\,dq$$

If a transformation of variables can be found that takes $\hat{\mathbf{u}}_{FU}\cdot\hat{\mathbf{d}}$ to $\sin(f')$, then it will be possible to use the result above to evaluate the integral. We know that

$$\hat{\mathbf{u}}_{FU} = \begin{bmatrix} c^+ \\ 0 \\ c^- \end{bmatrix} \qquad (6.29)$$

and

$$\hat{\mathbf{d}} = \begin{bmatrix} \cos(q)\cos(f) \\ \sin(q)\cos(f) \\ \sin(f) \end{bmatrix} \qquad (6.30)$$

so that

$$\hat{\mathbf{u}}_{FU}\cdot\hat{\mathbf{d}} = c^+\cos(q)\cos(f) + c^-\sin(f) \qquad (6.31)$$

The integrand is expressed in terms of polar coordinate angles (q, f), with which cartesian coordinates (x, y, z) may be associated. The transformed integrand will be expressed in terms of (q', f') or (x', y', z'). We require then that

$$\sin(f') = c^{+}\cos(q)\cos(f) + c^{-}\sin(f) \tag{6.32}$$

and, since $\sin(f')$ is by definition the cosine of the angle made with the z' axis, this may also be stated as

$$\hat{\mathbf{z}}' = \hat{\mathbf{u}}_{FU} \tag{6.33}$$

Expressing both in the original coordinate system, we therefore have

$$\hat{\mathbf{z}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{6.34}$$

and

$$\hat{\mathbf{z}}' = \begin{bmatrix} -c^{+} \\ 0 \\ c^{-} \end{bmatrix} \tag{6.35}$$

and it can be seen that the required coordinate transformation is a rotation about the y axis. A standard rotation matrix may therefore be used:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \mathbf{M}_{FU} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{6.36}$$

where

$$\mathbf{M}_{FU} = \begin{bmatrix} c^{-} & 0 & -c^{+} \\ 0 & 1 & 0 \\ c^{+} & 0 & c^{-} \end{bmatrix} \tag{6.37}$$

It may easily be verified that this is consistent with the requirements expressed by equations (6.32) and (6.33):

$$\begin{bmatrix} -c^{+} \\ 0 \\ c^{-} \end{bmatrix} = \begin{bmatrix} c^{-} & 0 & -c^{+} \\ 0 & 1 & 0 \\ c^{+} & 0 & c^{-} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{6.38}$$

Hence we have

$$\cos(q')\cos(f') = c^{-}\cos(q)\cos(f) - c^{+}\sin(f) \tag{6.39a}$$

$$\sin(q')\cos(f') = \sin(q)\cos(f) \tag{6.39b}$$

$$\sin(f') = c^{+}\cos(q)\cos(f) + c^{-}\sin(f) \tag{6.39c}$$

The integration now becomes

$$\int_{0}^{2p}\int_{-p/2}^{p/2} e^{jkr\hat{\mathbf{u}}_{FU}\cdot\hat{\mathbf{d}}}\cos(f)\,df\,dq = \int_{0}^{2p}\int_{-p/2}^{p/2} e^{jkr\sin(f')}\cos(f')\,df'\,dq' \tag{6.40}$$

or

$$L\left(e^{jkr\hat{\mathbf{u}}_{FU}\cdot\hat{\mathbf{d}}}\right) = L'\left(e^{jkr\sin(f')}\right) \tag{6.41}$$

where the notation

$$L'\left(f(q',f')\right) = \int_{0}^{2p}\int_{-p/2}^{p/2} f(q',f')\cos(f')\,df'\,dq' \tag{6.42}$$

indicates that we are working in the transformed coordinate system.

Evaluation of the transformed integral is straightforward:

$$\begin{aligned}\int_0^{2p}\int_{-p/2}^{p/2} e^{jkr\sin(f')}\cos(f')df'dq' &= \int_0^{2p} dq' \int_{-p/2}^{p/2} e^{jkr\sin(f')}\cos(f')df' \\ &= 2p\left[-\frac{j}{kr}e^{jkr\sin(f')}\right]_{-p/2}^{p/2} \\ &= -\frac{j2p}{kr}\left[e^{jkr}-e^{-jkr}\right] \\ &= \frac{4p}{kr}\frac{e^{jkr}-e^{-jkr}}{2j} \\ &= 4p\,\frac{\sin(kr)}{kr} \\ &= 4p\,j_0(kr)\end{aligned} \qquad (6.43)$$

Each of the eleven similar integrals evaluates to the same result. The transformation matrices employed are listed in Appendix 4.

We now consider the second set of integrals, such as

$$\int_0^{2p}\int_{-p/2}^{p/2} \hat{\mathbf{u}}_{FU}\cdot\hat{\mathbf{d}}e^{jkr\hat{\mathbf{u}}_{FU}\cdot\hat{\mathbf{d}}}\cos(f)df\,dq$$

Applying the same coordinate transformation yields

$$\int_0^{2p}\int_{-p/2}^{p/2} \hat{\mathbf{u}}_{FU}\cdot\hat{\mathbf{d}}e^{jkr\hat{\mathbf{u}}_{FU}\cdot\hat{\mathbf{d}}}\cos(f)df\,dq = \int_0^{2p}\int_{-p/2}^{p/2}\sin(f')e^{jkr\sin(f')}\cos(f')df'dq' \qquad (6.44)$$

Using the method of integration by parts, we obtain

$$\begin{aligned}\int\sin(f')e^{jkr\sin(f')}\cos(f')df' &= -\frac{j}{kr}e^{jkr\sin(f')}\sin(f')+\int\frac{j}{kr}e^{jkr\sin(f')}\cos(f')df' \\ &= -\frac{j}{kr}e^{jkr\sin(f')}\sin(f')+\frac{j}{kr}\int e^{jkr\sin(f')}\cos(f')df' \\ &= -\frac{j}{kr}e^{jkr\sin(f')}\sin(f')+\frac{j}{kr}\left(-\frac{j}{kr}e^{jkr\sin(f')}\right) \\ &= -\frac{j}{kr}e^{jkr\sin(f')}\sin(f')+\frac{1}{k^2r^2}e^{jkr\sin(f')} \\ &= e^{jkr\sin(f')}\left[\frac{1}{k^2r^2}-\frac{j}{kr}\sin(f')\right]\end{aligned} \qquad (6.45)$$

Hence,

$$\begin{aligned}\int_{-p/2}^{p/2}\sin(f')e^{jkr\sin(f')}\cos(f')df' &= \left[e^{jkr\sin(f')}\left(\frac{1}{k^2r^2}-\frac{j}{kr}\sin(f')\right)\right]_{-p/2}^{p/2}\\ &= e^{jkr}\left(\frac{1}{k^2r^2}-\frac{j}{kr}\right)-e^{-jkr}\left(\frac{1}{k^2r^2}+\frac{j}{kr}\right)\\ &= \frac{1}{k^2r^2}\left(e^{jkr}-e^{-jkr}\right)-\frac{j}{kr}\left(e^{jkr}+e^{-jkr}\right)\\ &= \frac{j2}{k^2r^2}\sin(kr)-\frac{j2}{kr}\cos(kr)\\ &= j2\left(\frac{\sin(kr)}{k^2r^2}-\frac{\cos(kr)}{kr}\right)\\ &= j2\,j_1(kr)\end{aligned} \tag{6.46}$$

and so

$$\begin{aligned}\int_0^{2p}\int_{-p/2}^{p/2}\sin(f')e^{jkr\sin(f')}\cos(f')df'dq' &= \int_0^{2p}dq'\int_{-p/2}^{p/2}\sin(f')e^{jkr\sin(f')}\cos(f')df'\\ &= j4p\, j_1(kr)\end{aligned} \tag{6.47}$$

Again, each of the eleven similar integrals evaluates to the same result. Substituting these results into equation (6.26) gives

$$\begin{aligned}A_0 &= aK_0\{4p\,[g_{FU}+g_{FD}+g_{BU}+g_{BD}+g_{LF}+g_{LB}\\ &\qquad +g_{RF}+g_{RB}+g_{UL}+g_{UR}+g_{DL}+g_{DR}]j_0(kr)\}\\ &\quad +bK_0\{j4p\,[g_{FU}+g_{FD}+g_{BU}+g_{BD}+g_{LF}+g_{LB}\\ &\qquad +g_{RF}+g_{RB}+g_{UL}+g_{UR}+g_{DL}+g_{DR}]j_1(kr)\}\\ &= \frac{G}{a+b}[g_{FU}+g_{FD}+g_{BU}+g_{BD}+g_{LF}+g_{LB}+g_{RF}+g_{RB}\\ &\qquad +g_{UL}+g_{UR}+g_{DL}+g_{DR}]\times[aj_0(kr)+jbj_1(kr)]\end{aligned} \tag{6.48}$$

Substantially the same method may be used to calculate the coefficients of the spherical harmonics of higher order, although there are some additional complications. To evaluate integrals such as

$$\int_0^{2\pi}\int_{-\pi/2}^{\pi/2} e^{jkr\hat{\mathbf{u}}_{FU}\cdot\hat{\mathbf{d}}}\cos(\theta)\cos(\phi)\cos(\phi)\,d\phi\,d\theta$$

it is necessary to express $\cos(\theta)\cos(\phi)$ (and in other cases $\sin(\theta)\cos(\phi)$ or $\sin(\phi)$) in terms of the transformed coordinate system. This may be accomplished by observing that if

$$\begin{bmatrix}\cos(\theta')\cos(\phi')\\ \sin(\theta')\cos(\phi')\\ \sin(\phi')\end{bmatrix} = \mathbf{M}\begin{bmatrix}\cos(\theta)\cos(\phi)\\ \sin(\theta)\cos(\phi)\\ \sin(\phi)\end{bmatrix} \tag{6.49}$$

then

$$\begin{bmatrix}\cos(\theta)\cos(\phi)\\ \sin(\theta)\cos(\phi)\\ \sin(\phi)\end{bmatrix} = \mathbf{M}^{-1}\begin{bmatrix}\cos(\theta')\cos(\phi')\\ \sin(\theta')\cos(\phi')\\ \sin(\phi')\end{bmatrix} \tag{6.50}$$

When computing the second-order coefficients, integrals such as

$$\int_0^{2\pi}\int_{-\pi/2}^{\pi/2} e^{jkr\hat{\mathbf{u}}_{FU}\cdot\hat{\mathbf{d}}}\cos(2\theta)\cos^2(\phi)\cos(\phi)\,d\phi\,d\theta$$

arise. Factors such as $\cos(2\theta)\cos^2(\phi)$ can be expanded, by application of trigonometric identities, into a polynomial in terms of $\cos(\theta)\cos(\phi)$, $\sin(\theta)\cos(\phi)$ and $\sin(\phi)$, which can then be transformed using inverse matrices as described above. Since a spherical harmonic is (by definition) a polynomial in the three direction cosines, such an expansion will always be possible for any spherical harmonic of any order.

The integrals which remain to be evaluated after these transformations have been performed become more complicated as the spherical harmonic order is increased. However, it so happens that each such integral can be evaluated by applying the method of integration by parts and using a previously found result. A list of the integrals which arise is given in Appendix 4.

The following expressions are obtained for the coefficients of the zeroth-order, first-order and second-order components of $\tilde{H}$:

$$A_0 = \frac{G}{a+b}[g_{FU} + g_{FD} + g_{BU} + g_{BD} + g_{LF} + g_{LB} + g_{RF} + g_{RB} + g_{UL} + g_{UR} + g_{DL} + g_{DR}]\times[aj_0(kr) + jbj_1(kr)] \quad (6.51a)$$

$$A_1 = \frac{G}{a+b}\left(c^+[g_{UL} + g_{UR} - g_{DL} - g_{DR}] + c^-[g_{FU} + g_{BU} - g_{FD} - g_{BD}]\right)\times[bj_0(kr) + j3aj_1(kr) - 2bj_2(kr)] \quad (6.51b)$$

$$A_{1,1} = \frac{G}{a+b}\left(c^+[g_{FU} + g_{FD} - g_{BU} - g_{BD}] + c^-[g_{LF} + g_{RF} - g_{LB} - g_{RB}]\right)\times[bj_0(kr) + j3aj_1(kr) - 2bj_2(kr)] \quad (6.51c)$$

$$B_{1,1} = \frac{G}{a+b}\left(c^+[g_{LF} + g_{LB} - g_{RF} - g_{RB}] + c^-[g_{UL} + g_{DL} - g_{UR} - g_{DR}]\right)\times[bj_0(kr) + j3aj_1(kr) - 2bj_2(kr)] \quad (6.51d)$$

$$A_2 = \frac{G}{a+b}\frac{1}{4\sqrt{5}}\left([\sqrt{5}-3][g_{FU} + g_{FD} + g_{BU} + g_{BD}] - [g_{LF} + g_{LB} + g_{RF} + g_{RB}] + [\sqrt{5}+3][g_{UL} + g_{UR} + g_{DL} + g_{DR}]\right)\times[j2bj_1(kr) - 5aj_2(kr) - j3bj_3(kr)] \quad (6.51e)$$

$$A_{2,1} = \frac{G}{a+b}\frac{1}{\sqrt{5}}[g_{FU} - g_{FD} - g_{BU} + g_{BD}]\times[j2bj_1(kr) - 5aj_2(kr) - j3bj_3(kr)] \quad (6.51f)$$

$$A_{2,2} = \frac{G}{a+b}\frac{1}{4}\left((c^+)^2[g_{FU} + g_{FD} + g_{BU} + g_{BD}] - \frac{1}{\sqrt{5}}[g_{LF} + g_{LB} + g_{RF} + g_{RB}] - (c^-)^2[g_{UL} + g_{UR} + g_{DL} + g_{DR}]\right)\times[j2bj_1(kr) - 5aj_2(kr) - j3bj_3(kr)] \quad (6.51g)$$

$$B_{2,1} = \frac{G}{a+b}\frac{1}{\sqrt{5}}[g_{UL} - g_{UR} - g_{DL} + g_{DR}]\times[j2bj_1(kr) - 5aj_2(kr) - j3bj_3(kr)] \quad (6.51h)$$

$$B_{2,2} = \frac{G}{a+b}\frac{1}{2\sqrt{5}}[g_{LF} - g_{LB} - g_{RF} + g_{RB}]\times[j2bj_1(kr) - 5aj_2(kr) - j3bj_3(kr)] \quad (6.51i)$$

Since these nine equations are not sufficient to determine the twelve matrix coefficients, we next consider the coefficients of the third-order spherical harmonics. Proceeding as before, we obtain:

$$A_3 = \frac{G}{a+b}\frac{1}{4}\Big(\big[1+\sqrt{5}\big]c^-[g_{FU}+g_{BU}-g_{FD}-g_{BD}] - \big[\sqrt{5}-1\big]c^+[g_{UL}+g_{UR}-g_{DL}-g_{DR}]\Big) \times\big[3bj_2(kr)+j7aj_3(kr)-4bj_4(kr)\big] \tag{6.52a}$$

$$A_{3,1} = \frac{G}{a+b}\frac{1}{8}\Big(\big[\sqrt{5}-3\big]c^+[g_{FU}+g_{FD}-g_{BU}-g_{BD}] + 2c^-[g_{LF}+g_{RF}-g_{LB}-g_{RB}]\Big) \times\big[3bj_2(kr)+j7aj_3(kr)-4bj_4(kr)\big] \tag{6.52b}$$

$$A_{3,2} = \frac{G}{a+b}\frac{1}{8\sqrt{5}}\Big(\big[1+\sqrt{5}\big]c^-[g_{FD}+g_{BD}-g_{FU}-g_{BU}] - \big[\sqrt{5}-1\big]c^+[g_{DL}+g_{DR}-g_{UL}-g_{UR}]\Big) \times\big[3bj_2(kr)+j7aj_3(kr)-4bj_4(kr)\big] \tag{6.52c}$$

$$A_{3,3} = \frac{G}{a+b}\frac{\sqrt{5}}{240}\Big(\big[1+\sqrt{5}\big]c^+[g_{BU}+g_{BD}-g_{FU}-g_{FD}] + 2\big[2+\sqrt{5}\big]c^-[g_{LF}+g_{RF}-g_{LB}-g_{RB}]\Big) \times\big[3bj_2(kr)+j7aj_3(kr)-4bj_4(kr)\big] \tag{6.52d}$$

$$B_{3,1} = \frac{G}{a+b}\frac{1}{8}\Big(2c^+[g_{LF}+g_{LB}-g_{RF}-g_{RB}] + \big[3+\sqrt{5}\big]c^-[g_{UR}+g_{DR}-g_{UL}-g_{DL}]\Big) \times\big[3bj_2(kr)+j7aj_3(kr)-4bj_4(kr)\big] \tag{6.52e}$$

$$B_{3,2} = 0 \tag{6.52f}$$

$$B_{3,3} = \frac{G}{a+b}\frac{\sqrt{5}}{240}\Big(2\big[\sqrt{5}-2\big]c^+[g_{RF}+g_{RB}-g_{LF}-g_{LB}] + \big[\sqrt{5}-1\big]c^-[g_{UL}+g_{DL}-g_{UR}-g_{DR}]\Big) \times\big[3bj_2(kr)+j7aj_3(kr)-4bj_4(kr)\big] \tag{6.52g}$$

In total, we now have fifteen equations (excluding that involving $B_{3,2}$, which is independent of the matrix coefficients). However, the third-order Laplace series coefficients are not all linearly independent; it may be observed that

$$A_3 = -2\sqrt{5}\,A_{3,2} \tag{6.53a}$$

$$A_{3,1} = 6\big(5-2\sqrt{5}\big)A_{3,3} \tag{6.53b}$$

$$B_{3,1} = -6\big(5+2\sqrt{5}\big)B_{3,3} \tag{6.53c}$$

Hence, we have obtained exactly twelve linearly independent equations in the twelve A-B matrix coefficients; these can therefore now be determined. For each B-format signal, we set the coefficient of the appropriate spherical harmonic to a suitable value, and all other coefficients to zero; we then solve the resulting set of equations for the twelve matrix coefficients. Note that, while this “suitable value” is 1 for the zeroth-order and first-order component signals, it takes different values for the second-order signals because of the scaling factors which appear in the definitions of the second-order spherical harmonics. For example, the coefficient $B_{2,2}$ is associated with the spherical harmonic $3\sin(2q)\cos^2(f)$; hence, to obtain the desired polar response, $B_{2,2}$ must be set equal to $\frac{1}{3}$.

It is desired that the A-B matrix should be frequency-independent, as it is for the first-order soundfield microphone; hence, the frequency response functions are omitted at this stage. For convenience we also disregard the $G/(a+b)$ factor. The A-B matrix derived therefore depends only on the geometry of the array; the polar responses of the individual capsules will be taken into account when designing the non-coincidence compensation filters.

The matrix coefficients obtained are shown in table 6.1.

Note that each of the signals *S*, *T* and *V* is obtained from a rectangular arrangement of capsules similar to that described in section 3.6.

	W	X	Y	Z	R	S	T	U	V
g_{FU}	$\frac{1}{12}$	$\frac{1}{4}c^{+}$	0	$\frac{1}{4}c^{-}$	$\frac{\sqrt{5}}{48}(\sqrt{5}-3)$	$\frac{\sqrt{5}}{6}$	0	$\frac{\sqrt{5}}{24}(1+\sqrt{5})$	0
g_{FD}	$\frac{1}{12}$	$\frac{1}{4}c^{+}$	0	$-\frac{1}{4}c^{-}$	$\frac{\sqrt{5}}{48}(\sqrt{5}-3)$	$-\frac{\sqrt{5}}{6}$	0	$\frac{\sqrt{5}}{24}(1+\sqrt{5})$	0
g_{BU}	$\frac{1}{12}$	$-\frac{1}{4}c^{+}$	0	$\frac{1}{4}c^{-}$	$\frac{\sqrt{5}}{48}(\sqrt{5}-3)$	$-\frac{\sqrt{5}}{6}$	0	$\frac{\sqrt{5}}{24}(1+\sqrt{5})$	0
g_{BD}	$\frac{1}{12}$	$-\frac{1}{4}c^{+}$	0	$-\frac{1}{4}c^{-}$	$\frac{\sqrt{5}}{48}(\sqrt{5}-3)$	$\frac{\sqrt{5}}{6}$	0	$\frac{\sqrt{5}}{24}(1+\sqrt{5})$	0
g_{LF}	$\frac{1}{12}$	$\frac{1}{4}c^{-}$	$\frac{1}{4}c^{+}$	0	$-\frac{5}{24}$	0	0	$-\frac{\sqrt{5}}{12}$	$\frac{\sqrt{5}}{6}$
g_{LB}	$\frac{1}{12}$	$-\frac{1}{4}c^{-}$	$\frac{1}{4}c^{+}$	0	$-\frac{5}{24}$	0	0	$-\frac{\sqrt{5}}{12}$	$-\frac{\sqrt{5}}{6}$
g_{RF}	$\frac{1}{12}$	$\frac{1}{4}c^{-}$	$-\frac{1}{4}c^{+}$	0	$-\frac{5}{24}$	0	0	$-\frac{\sqrt{5}}{12}$	$-\frac{\sqrt{5}}{6}$
g_{RB}	$\frac{1}{12}$	$-\frac{1}{4}c^{-}$	$-\frac{1}{4}c^{+}$	0	$-\frac{5}{24}$	0	0	$-\frac{\sqrt{5}}{12}$	$\frac{\sqrt{5}}{6}$
g_{UL}	$\frac{1}{12}$	0	$\frac{1}{4}c^{-}$	$\frac{1}{4}c^{+}$	$\frac{\sqrt{5}}{48}(\sqrt{5}+3)$	0	$\frac{\sqrt{5}}{6}$	$\frac{\sqrt{5}}{24}(1-\sqrt{5})$	0
g_{UR}	$\frac{1}{12}$	0	$-\frac{1}{4}c^{-}$	$\frac{1}{4}c^{+}$	$\frac{\sqrt{5}}{48}(\sqrt{5}+3)$	0	$-\frac{\sqrt{5}}{6}$	$\frac{\sqrt{5}}{24}(1-\sqrt{5})$	0
g_{DL}	$\frac{1}{12}$	0	$\frac{1}{4}c^{-}$	$-\frac{1}{4}c^{+}$	$\frac{\sqrt{5}}{48}(\sqrt{5}+3)$	0	$-\frac{\sqrt{5}}{6}$	$\frac{\sqrt{5}}{24}(1-\sqrt{5})$	0
g_{DR}	$\frac{1}{12}$	0	$-\frac{1}{4}c^{-}$	$-\frac{1}{4}c^{+}$	$\frac{\sqrt{5}}{48}(\sqrt{5}+3)$	0	$\frac{\sqrt{5}}{6}$	$\frac{\sqrt{5}}{24}(1-\sqrt{5})$	0

Table 6.1: A-B Matrix Coefficients for Second-Order Soundfield Microphone

6.3: Presence of Unwanted Spherical Harmonic Components

Using the methods described in the previous section, it is possible to obtain expressions in terms of the A-B matrix coefficients for higher order spherical harmonic components of $\tilde{H}$. Since these matrix coefficients are now known for each of the B-format signals, it is possible to establish the presence in each of the B-format signals of unwanted spherical harmonic components.

It is convenient to define the functions

$$\begin{aligned} И_n(n,a) &= j^{(1-n)}\left[n(1-n)\,j_{n-1}(a) + j(2n+1)n\,j_n(a) - (n+1)(1-n)\,j_{n+1}(a)\right] \\ &= j^{-n}\left[j(1-n)\left(nj_{n-1}(a) - (n+1)\,j_{n+1}(a)\right) - (2n+1)n\,j_n(a)\right] \\ &= j^{-n}(2n+1)\left[j(1-n)\frac{dj_n(a)}{da} - n\,j_n(a)\right] \end{aligned} \tag{6.54}$$

and to introduce normalised capsule polar pattern constants

$$a' = \frac{a}{a+b} \tag{6.55a}$$

$$\begin{aligned} b' &= \frac{b}{a+b} \\ &= 1 - a' \end{aligned} \tag{6.55b}$$

None of the signals contains third-order spherical harmonic components, since these were eliminated as part of the A-B matrix design procedure. The fourth-order Laplace series coefficients are

$$\begin{aligned} A_4 = G\frac{1}{16}\Big\{&-\left(3+\sqrt{5}\right)\left[g_{FU} + g_{FD} + g_{BU} + g_{BD}\right] - 6\left[g_{LF} + g_{LB} + g_{RF} + g_{RB}\right] \\ &+\left(3-\sqrt{5}\right)\left[g_{UL} + g_{UR} + g_{DL} + g_{DR}\right]\Big\} \times И_4(a', kr) \end{aligned} \tag{6.56a}$$

$$A_{4,1} = G\frac{7-\sqrt{5}}{40}\left[g_{FU} - g_{FD} - g_{BU} + g_{BD}\right] \times И_4(a', kr) \tag{6.56b}$$

$$\begin{aligned} A_{4,2} = G\frac{1}{240}\Big\{&-\left(9-\sqrt{5}\right)\left[g_{FU} + g_{FD} + g_{BU} + g_{BD}\right] - 2\sqrt{5}\left[g_{LF} + g_{LB} + g_{RF} + g_{RB}\right] \\ &+\left(9+\sqrt{5}\right)\left[g_{UL} + g_{UR} + g_{DL} + g_{DR}\right]\Big\} \times И_4(a', kr) \end{aligned} \tag{6.56c}$$

$$A_{4,3} = G\frac{1+\sqrt{5}}{240}\left[-g_{FU} + g_{FD} + g_{BU} - g_{BD}\right] \times И_4(a', kr) \tag{6.56d}$$

$$\begin{aligned} A_{4,4} = G\frac{1}{1920}\Big\{&-\left(3+\sqrt{5}\right)\left[g_{FU} + g_{FD} + g_{BU} + g_{BD}\right] + 6\left[g_{LF} + g_{LB} + g_{RF} + g_{RB}\right] \\ &-\left(3-\sqrt{5}\right)\left[g_{UL} + g_{UR} + g_{DL} + g_{DR}\right]\Big\} \times И_4(a', kr) \end{aligned} \tag{6.56e}$$

$$B_{4,1} = G\frac{7+\sqrt{5}}{40}\left[-g_{UL} + g_{UR} + g_{DL} - g_{DR}\right] \times И_4(a', kr) \tag{6.56f}$$

$$B_{4,2} = G\frac{\sqrt{5}}{60}\left[g_{LF} - g_{LB} - g_{RF} + g_{RB}\right] \times И_4(a', kr) \tag{6.56g}$$

$$B_{4,3} = G\frac{\sqrt{5}-1}{240}\left[g_{UL} - g_{UR} - g_{DL} + g_{DR}\right]\times И_4(a',kr) \tag{6.56h}$$

$$B_{4,4} = G\frac{1}{240}\left[g_{LF} - g_{LB} - g_{RF} + g_{RB}\right]\times И_4(a',kr) \tag{6.56i}$$

The fifth-order coefficients are given by

$$A_5 = G\frac{\sqrt{5}}{400}\left\{\left(35-11\sqrt{5}\right)c^-\left[g_{FU} - g_{FD} + g_{BU} - g_{BD}\right] + \left(35+11\sqrt{5}\right)c^+\left[-g_{UL} - g_{UR} + g_{DL} + g_{DR}\right]\right\}\times И_5(a',kr) \tag{6.57a}$$

$$A_{5,1} = G\frac{\sqrt{5}}{400}\left\{\left(35-3\sqrt{5}\right)c^+\left[-g_{FU} - g_{FD} + g_{BU} + g_{BD}\right] + 10\sqrt{5}c^-\left[g_{LF} - g_{LB} + g_{RF} - g_{RB}\right]\right\}\times И_5(a',kr) \tag{6.57b}$$

$$A_{5,2} = G\frac{1}{80}\left\{\left(\sqrt{5}-1\right)c^-\left[-g_{FU} + g_{FD} - g_{BU} + g_{BD}\right] + \left(\sqrt{5}+1\right)c^+\left[-g_{UL} - g_{UR} + g_{DL} + g_{DR}\right]\right\}\times И_5(a',kr) \tag{6.57c}$$

$$A_{5,3} = G\frac{1}{1920}\left\{\left(13-\sqrt{5}\right)c^+\left[g_{FU} + g_{FD} - g_{BU} - g_{BD}\right] + 2\sqrt{5}\left(2+\sqrt{5}\right)c^-\left[g_{LF} - g_{LB} + g_{RF} - g_{RB}\right]\right\}\times И_5(a',kr) \tag{6.57d}$$

$$A_{5,4} = G\frac{1}{1920}\left\{\left(3+\sqrt{5}\right)c^-\left[g_{FU} - g_{FD} + g_{BU} - g_{BD}\right] + \left(3-\sqrt{5}\right)c^+\left[g_{UL} + g_{UR} - g_{DL} - g_{DR}\right]\right\}\times И_5(a',kr) \tag{6.57e}$$

$$A_{5,5} = G\frac{1}{19200}\left\{\left(3+\sqrt{5}\right)c^+\left[g_{FU} + g_{FD} - g_{BU} - g_{BD}\right] + 2\left(2\sqrt{5}-1\right)c^-\left[g_{LF} - g_{LB} + g_{RF} - g_{RB}\right]\right\}\times И_5(a',kr) \tag{6.57f}$$

$$B_{5,1} = G\frac{\sqrt{5}}{400}\left\{10\sqrt{5}c^+\left[g_{LF} + g_{LB} - g_{RF} - g_{RB}\right] + \left(35+3\sqrt{5}\right)c^-\left[g_{UL} - g_{UR} + g_{DL} - g_{DR}\right]\right\}\times И_5(a',kr) \tag{6.57g}$$

$$B_{5,2} = 0 \tag{6.57h}$$

$$B_{5,3} = G\frac{1}{1920}\left\{2\sqrt{5}\left(\sqrt{5}-2\right)c^+\left[-g_{LF} - g_{LB} + g_{RF} + g_{RB}\right] + \left(13+\sqrt{5}\right)c^-\left[-g_{UL} + g_{UR} - g_{DL} + g_{DR}\right]\right\}\times И_5(a',kr) \tag{6.57i}$$

$$B_{5,4} = 0 \tag{6.57j}$$

$$B_{5,5} = G\frac{1}{19200}\left\{2\left(2\sqrt{5}+1\right)c^{+}\left[-g_{LF}-g_{LB}+g_{RF}+g_{RB}\right] + \left(3-\sqrt{5}\right)c^{-}\left[g_{UL}-g_{UR}+g_{DL}-g_{DR}\right]\right\}\times И_{5}(a',kr) \tag{6.57k}$$

The sixth-order Laplace series coefficients are given by

$$A_{6} = G\frac{1}{400}\left\{-\left(63\sqrt{5}-20\right)\left[g_{FU}+g_{FD}+g_{BU}+g_{BD}\right] +125\left[g_{LF}+g_{LB}+g_{RF}+g_{RB}\right] +\left(20+63\sqrt{5}\right)\left[g_{UL}+g_{UR}+g_{DL}+g_{DR}\right]\right\}\times И_{6}(a',kr) \tag{6.58a}$$

$$A_{6,1} = G\frac{15+\sqrt{5}}{400}\left[g_{FU}-g_{FD}-g_{BU}+g_{BD}\right]\times И_{6}(a',kr) \tag{6.58b}$$

$$A_{6,2} = G\frac{\sqrt{5}}{1600}\left\{\left(14-\sqrt{5}\right)\left[g_{FU}+g_{FD}+g_{BU}+g_{BD}\right] +5\left[g_{LF}+g_{LB}+g_{RF}+g_{RB}\right] +\left(14+\sqrt{5}\right)\left[g_{UL}+g_{UR}+g_{DL}+g_{DR}\right]\right\}\times И_{6}(a',kr) \tag{6.58c}$$

$$A_{6,3} = G\frac{7\sqrt{5}-15}{9600}\left[-g_{FU}+g_{FD}+g_{BU}-g_{BD}\right]\times И_{6}(a',kr) \tag{6.58d}$$

$$A_{6,4} = G\frac{1}{48000}\left\{-\left(20+3\sqrt{5}\right)\left[g_{FU}+g_{FD}+g_{BU}+g_{BD}\right] -15\left[g_{LF}+g_{LB}+g_{RF}+g_{RB}\right] -\left(20-3\sqrt{5}\right)\left[g_{UL}+g_{UR}+g_{DL}+g_{DR}\right]\right\}\times И_{6}(a',kr) \tag{6.58e}$$

$$A_{6,5} = G\frac{5+3\sqrt{5}}{96000}\left[-g_{FU}+g_{FD}+g_{BU}-g_{BD}\right]\times И_{6}(a',kr) \tag{6.58f}$$

$$A_{6,6} = G\frac{\sqrt{5}}{576000}\left\{-\left(\sqrt{5}+2\right)\left[g_{FU}+g_{FD}+g_{BU}+g_{BD}\right] -11\left[g_{LF}+g_{LB}+g_{RF}+g_{RB}\right] +\left(\sqrt{5}-2\right)\left[g_{UL}+g_{UR}+g_{DL}+g_{DR}\right]\right\}\times И_{6}(a',kr) \tag{6.58g}$$

$$B_{6,1} = \frac{15-\sqrt{5}}{400}\left[-g_{UL}+g_{UR}+g_{DL}-g_{DR}\right]\times И_{6}(a',kr) \tag{6.58h}$$

$$B_{6,2} = \frac{\sqrt{5}}{160}\left[-g_{LF}+g_{LB}+g_{RF}-g_{RB}\right]\times И_{6}(a',kr) \tag{6.58i}$$

$$B_{6,3} = \frac{15+7\sqrt{5}}{9600}\left[g_{UL}-g_{UR}-g_{DL}+g_{DR}\right]\times И_{6}(a',kr) \tag{6.58j}$$

$$B_{6,4} = \frac{1}{2400}\left[-g_{LF} + g_{LB} + g_{RF} - g_{RB}\right]\times И_6(a',kr) \tag{6.58k}$$

$$B_{6,5} = \frac{3\sqrt{5}-5}{96000}\left[-g_{UL} + g_{UR} + g_{DL} - g_{DR}\right]\times И_6(a',kr) \tag{6.58l}$$

$$B_{6,6} = \frac{\sqrt{5}}{288000}\left[g_{LF} - g_{LB} - g_{RF} + g_{RB}\right]\times И_6(a',kr) \tag{6.58m}$$

From these general expressions, the Laplace series coefficients for each of the B-format signals are obtained by substituting in the known values for the A-B matrix coefficients. It so happens that the majority of the coefficients in the Laplace series for each signal are zero; only the non-zero coefficients are listed here.

For $\tilde{W}$:

$$A_6\left\{\tilde{W}\right\} = \frac{11}{80} GИ_6(a',kr) \tag{6.59a}$$

$$A_{6,2}\left\{\tilde{W}\right\} = \frac{11\sqrt{5}}{1600} GИ_6(a',kr) \tag{6.59b}$$

$$A_{6,4}\left\{\tilde{W}\right\} = -\frac{11}{28800} GИ_6(a',kr) \tag{6.59c}$$

$$A_{6,6}\left\{\tilde{W}\right\} = -\frac{\sqrt{5}}{115200} GИ_6(a',kr) \tag{6.59d}$$

For $\tilde{X}$:

$$A_{5,1}\left\{\tilde{X}\right\} = -\frac{21\sqrt{5}-15}{400} GИ_5(a',kr) \tag{6.60a}$$

$$A_{5,3}\left\{\tilde{X}\right\} = \frac{15+3\sqrt{5}}{3200} GИ_5(a',kr) \tag{6.60b}$$

$$A_{5,5}\left\{\tilde{X}\right\} = \frac{3\sqrt{5}-1}{19200} GИ_5(a',kr) \tag{6.60c}$$

For $\tilde{Y}$:

$$B_{5,1}\left\{\tilde{Y}\right\} = \frac{21\sqrt{5}+15}{400} GИ_5(a',kr) \tag{6.61a}$$

$$B_{5,3}\{\tilde{Y}\} = -\frac{15-3\sqrt{5}}{3200} G\text{И}_5(a', kr) \quad (6.61b)$$

$$B_{5,5}\{\tilde{Y}\} = -\frac{3\sqrt{5}+1}{19200} G\text{И}_5(a', kr) \quad (6.61c)$$

For $\tilde{Z}$:

$$A_5\{\tilde{Z}\} = -\frac{9}{40} G\text{И}_5(a', kr) \quad (6.62a)$$

$$A_{5,2}\{\tilde{Z}\} = -\frac{3\sqrt{5}}{200} G\text{И}_5(a', kr) \quad (6.62b)$$

$$A_{5,4}\{\tilde{Z}\} = \frac{1}{960} G\text{И}_5(a', kr) \quad (6.62c)$$

For $\tilde{R}$:

$$A_4\{\tilde{R}\} = -\frac{5}{16} G\text{И}_4(a', kr) \quad (6.63a)$$

$$A_{4,2}\{\tilde{R}\} = \frac{7\sqrt{5}}{240} G\text{И}_4(a', kr) \quad (6.63b)$$

$$A_{4,4}\{\tilde{R}\} = -\frac{1}{384} G\text{И}_4(a', kr) \quad (6.63c)$$

$$A_6\{\tilde{R}\} = \frac{21}{120} G\text{И}_6(a', kr) \quad (6.63d)$$

$$A_{6,2}\{\tilde{R}\} = \frac{3\sqrt{5}}{480} G\text{И}_6(a', kr) \quad (6.63e)$$

$$A_{6,4}\{\tilde{R}\} = \frac{1}{14400} G\text{И}_6(a', kr) \quad (6.63f)$$

$$A_{6,6}\{\tilde{R}\} = \frac{\sqrt{5}}{57600} G\text{И}_6(a', kr) \quad (6.63g)$$

For $\tilde{S}$:

$$A_{4,1}\{\tilde{S}\} = \frac{7\sqrt{5}-5}{60} G\text{И}_4(a', kr) \quad (6.64a)$$

$$A_{4,3}\{\tilde{S}\} = -\frac{5+\sqrt{5}}{360} G\text{И}_4(a', kr) \quad (6.64b)$$

$$A_{6,1}\{\tilde{S}\} = \frac{3\sqrt{5}+1}{120} G\text{И}_6(a', kr) \quad (6.64c)$$

$$A_{6,3}\{\tilde{S}\} = -\frac{7-3\sqrt{5}}{2880} G\text{И}_6(a', kr) \tag{6.64d}$$

$$A_{6,5}\{\tilde{S}\} = -\frac{3+\sqrt{5}}{28800} G\text{И}_6(a', kr) \tag{6.64e}$$

For $\tilde{T}$:

$$B_{4,1}\{\tilde{T}\} = -\frac{7\sqrt{5}+5}{60} G\text{И}_4(a', kr) \tag{6.65a}$$

$$B_{4,3}\{\tilde{T}\} = \frac{5-\sqrt{5}}{360} G\text{И}_4(a', kr) \tag{6.65b}$$

$$B_{6,1}\{\tilde{T}\} = -\frac{3\sqrt{5}-1}{120} G\text{И}_6(a', kr) \tag{6.65c}$$

$$B_{6,3}\{\tilde{T}\} = \frac{7+3\sqrt{5}}{2880} G\text{И}_6(a', kr) \tag{6.65d}$$

$$B_{6,5}\{\tilde{T}\} = -\frac{3-\sqrt{5}}{28800} G\text{И}_6(a', kr) \tag{6.65e}$$

For $\tilde{U}$:

$$A_{4}\{\tilde{U}\} = \frac{7\sqrt{5}}{24} G\text{И}_4(a', kr) \tag{6.66a}$$

$$A_{4,2}\{\tilde{U}\} = -\frac{1}{24} G\text{И}_4(a', kr) \tag{6.66b}$$

$$A_{4,4}\{\tilde{U}\} = -\frac{7\sqrt{5}}{2880} G\text{И}_4(a', kr) \tag{6.66c}$$

$$A_{6}\{\tilde{U}\} = -\frac{7\sqrt{5}}{20} G\text{И}_6(a', kr) \tag{6.66d}$$

$$A_{6,2}\{\tilde{U}\} = \frac{1}{240} G\text{И}_6(a', kr) \tag{6.66e}$$

$$A_{6,4}\{\tilde{U}\} = -\frac{\sqrt{5}}{7200} G\text{И}_6(a', kr) \tag{6.66f}$$

$$A_{6,6}\{\tilde{U}\} = \frac{1}{86400} G\text{И}_6(a', kr) \tag{6.66g}$$

For $\tilde{V}$:

$$B_{4,2}\{\tilde{V}\} = \frac{1}{18} G И_4(a', kr) \tag{6.67a}$$

$$B_{4,4}\{\tilde{V}\} = \frac{\sqrt{5}}{360} G И_4(a', kr) \tag{6.67b}$$

$$B_{6,2}\{\tilde{V}\} = -\frac{1}{48} G И_6(a', kr) \tag{6.67c}$$

$$B_{6,4}\{\tilde{V}\} = -\frac{\sqrt{5}}{3600} G И_6(a', kr) \tag{6.67d}$$

$$B_{6,6}\{\tilde{V}\} = \frac{1}{86400} G И_6(a', kr) \tag{6.67e}$$

From these results it can be seen that $\tilde{W}$ is corrupted by unwanted spherical harmonics of order six; the first-order component signals are corrupted by spurious fifth-order spherical harmonics; and the second-order signals contain both fourth-order and sixth-order unwanted spherical harmonic components.

In Chapter 5, it was noted that the B-format signals obtained from the first-order soundfield microphone are contaminated by spurious second-order or third-order spherical harmonics. The second-order soundfield microphone therefore represents a considerable improvement in this respect. Since the higher order spherical harmonics have coefficients which depend on higher order spherical Bessel functions, which in turn remain small up until greater values of *kr*, so it may be expected that the maximum frequency to which effective coincidence is maintained will exceed that given by equation (5.9). This advantage will, however, probably be opposed to some extent by the fact that the array radius is likely to larger for a second-order soundfield microphone.

6.4: Non-Coincidence Compensation Filtering

The author has not considered the design of non-coincidence compensation filters in detail; the design of practical approximations to the theoretically ideal characteristics is a matter of practical implementation rather than fundamental theory, and therefore outside the scope of this thesis. Nevertheless, the following observations can be made.

By substituting the coefficients given in table 6.1 into equation (6.51), we obtain the following results:

$$A_0\{\tilde{W}\} = G[a'\,j_0(kr) + jb'\,j_1(kr)] \tag{6.68a}$$

$$\begin{aligned} A_1\{\tilde{Z}\} &= G[b'\,j_0(kr) + j3a'\,j_1(kr) - 2b'\,j_2(kr)] \\ &= G\text{И}_1(a', kr) \end{aligned} \tag{6.68b}$$

$$\begin{aligned} A_{1,1}\{\tilde{X}\} &= G[b'\,j_0(kr) + j3a'\,j_1(kr) - 2b'\,j_2(kr)] \\ &= G\text{И}_1(a', kr) \end{aligned} \tag{6.68c}$$

$$\begin{aligned} B_{1,1}\{\tilde{Y}\} &= G[b'\,j_0(kr) + j3a'\,j_1(kr) - 2b'\,j_2(kr)] \\ &= G\text{И}_1(a', kr) \end{aligned} \tag{6.68d}$$

$$\begin{aligned} A_2\{\tilde{R}\} &= G[j2b'\,j_1(kr) - 5a'\,j_2(kr) - j3b'\,j_3(kr)] \\ &= -G\text{И}_2(a', kr) \end{aligned} \tag{6.68e}$$

$$\begin{aligned} A_{2,1}\{\tilde{S}\} &= G\frac{2}{3}[j2b'\,j_1(kr) - 5a'\,j_2(kr) - j3b'\,j_3(kr)] \\ &= -G\frac{2}{3}\text{И}_2(a', kr) \end{aligned} \tag{6.68f}$$

$$\begin{aligned} A_{2,2}\{\tilde{U}\} &= G\frac{1}{3}[j2b'\,j_1(kr) - 5a'\,j_2(kr) - j3b'\,j_3(kr)] \\ &= -G\frac{1}{3}\text{И}_2(a', kr) \end{aligned} \tag{6.68g}$$

$$\begin{aligned} B_{2,1}\{\tilde{T}\} &= G\frac{2}{3}[j2b'\,j_1(kr) - 5a'\,j_2(kr) - j3b'\,j_3(kr)] \\ &= -G\frac{2}{3}\text{И}_2(a', kr) \end{aligned} \tag{6.68h}$$

$$\begin{aligned} B_{2,2}\{\tilde{V}\} &= G\frac{1}{3}[j2b'\,j_1(kr) - 5a'\,j_2(kr) - j3b'\,j_3(kr)] \\ &= -G\frac{1}{3}\text{И}_2(a', kr) \end{aligned} \tag{6.68i}$$

The factors of $\frac{2}{3}$ and $\frac{1}{3}$ in the expressions for $A_{2,1}$, $A_{2,2}$, $B_{2,1}$, and $B_{2,2}$ are due to the factors of $\frac{3}{2}$ and 3 which appear in the definitions of the corresponding spherical harmonics (see page 128); they therefore do not imply the need for compensatory scaling of the signals.

The impulse responses corresponding to these frequency response functions may be found by taking the inverse Fourier transforms. Let

$$t = \frac{r}{c} \tag{6.69}$$

then

$$kr = tw \tag{6.70}$$

and employing the inverse Fourier transforms of the spherical Bessel functions developed in Chapter 2, we obtain

$$\begin{aligned} D_0(t) &= \hat{F}^{-1}\{G[a'\,j_0(tw) + jb'\,j_1(tw)]\} \\ &= \frac{G}{2t^2}\left(-b't + a't\right) r_t(t) \end{aligned} \tag{6.71a}$$

$$\begin{aligned} D_1(t) &= \hat{F}^{-1}\{G\text{И}_1(a', tw)\} \\ &= \frac{3G}{2t^3}\left(b't^2 - a't\,t\right) r_t(t) \end{aligned} \tag{6.71b}$$

$$\begin{aligned} D_2(t) &= \hat{F}^{-1}\{-G\text{И}_2(a', tw)\} \\ &= \frac{5G}{4t^4}\left(-3b't^3 + 3a't\,t^2 + b't^2 t - a't^3\right) r_t(t) \end{aligned} \tag{6.71c}$$

where

$$r_t(t) = \begin{cases} 1 & -t < t < t \\ 0 & \text{otherwise} \end{cases} \tag{6.72}$$

It may be noted that the frequency responses of the desired spherical harmonic components of the zeroth-order and first-order signals have the same form as in the case of the first-order soundfield microphone. Filters that have proved to give acceptable results with the first-order soundfield microphone might well therefore be equally suitable for use with the second-order microphone.

In the case of the second-order signals, the required filtering is fundamentally different in one respect. From equation (2.14), it can be seen that the frequency response function for the second-order spherical harmonic components becomes zero for $kr = 0$. This is because we are approximating second-order directional derivatives by taking the difference between the outputs of first-order microphone capsules. An integration with respect to time is therefore necessary, not to compensate for the spacing of the capsules, but as a fundamental part of the method being utilised to obtain the signals. The filtering will therefore serve a dual purpose so far as these signals are concerned, since at higher frequencies compensation for the effects of the capsule spacing will still be required.

It must be noted that suitable filters cannot be designed on the basis of theory alone. During the development of the first-order soundfield microphone, it was found that although filters developed from theoretical analysis gave a substantial improvement over arrays without filtering, to obtain optimum performance it was necessary to take into account experimental information [7]. Certainly one expects that this will be the case with the second-order soundfield microphone as well, since there will inevitably be departures from ideal behaviour which are not represented in the theoretical treatment.

6.5: Additional B-Format Signal Processing

6.5.1: Rotation & Elevation

The rotation and elevation controls for the second-order soundfield microphone must clearly have an identical effect on the zeroth-order and first-order signals as in the case of the first-order microphone.

By trigonometric manipulation it may be established that the rotation control modifies the second-order component signals as follows:

$$R_2 = R_1 \quad (6.73a)$$

$$S_2 = \cos(q)S_1 + \sin(q)T_1 \quad (6.73b)$$

$$T_2 = -\sin(q)S_1 + \cos(q)T_1 \quad (6.73c)$$

$$U_2 = \cos(2q)U_1 + \sin(2q)V_1 \quad (6.73d)$$

$$V_2 = -\sin(2q)U_1 + \cos(2q)V_1 \quad (6.73e)$$

The effect of the elevation control on the second-order signals may similarly be established to be:

$$R_2 = \tfrac{1}{4}\left(1 + 3\cos(2f)\right)R_1 - \tfrac{3}{4}\sin(2f)S_1 + \tfrac{3}{8}\left(1 - \cos(2f)\right)U_1 \quad (6.74a)$$

$$S_2 = \sin(2f)R_1 + \cos(2f)S_1 - \tfrac{1}{2}\sin(2f)U_1 \quad (6.74b)$$

$$T_2 = \cos(f)T_1 - \sin(f)V_1 \quad (6.74c)$$

$$U_2 = \tfrac{1}{2}\left(1 - \cos(2f)\right)R_1 + \tfrac{1}{2}\sin(2f)S_1 + \tfrac{1}{4}\left(3 + \cos(2f)\right)U_1 \quad (6.74d)$$

$$V_2 = \sin(f)T_1 + \cos(f)V_1 \quad (6.74e)$$

6.5.2: Side-Fire / End-Fire Switching & Inversion

The compensatory signal processing required to facilitate end-fire operation is:

$$W_2 = W_1 \quad (6.75a)$$

$$X_2 = Z_1 \quad (6.75b)$$

$$Y_2 = Y_1 \quad (6.75c)$$

$$Z_2 = -X_1 \quad (6.75d)$$

$$R_2 = -\tfrac{1}{2}R_1 + \tfrac{3}{4}U_1 \quad (6.75e)$$

$$S_2 = -S_1 \quad (6.75f)$$

$$T_2 = V_1 \quad (6.75g)$$

$$U_2 = R_1 + \tfrac{1}{2}U_1 \quad (6.75h)$$

$$V_2 = T_1 \quad (6.75i)$$

As in the case of the first-order soundfield microphone, inverted operation requires only a polarity reversal of some of the B-format signals:

$$Y_2 = -Y_1 \quad (6.76a)$$

$$Z_2 = -Z_1 \quad (6.76b)$$

$$S_2 = -S_1 \quad (6.76c)$$

$$T_2 = -T_1 \quad (6.76d)$$

$$V_2 = -V_1 \quad (6.76e)$$

6.5.3: Dominance

The author has proved that it is not possible to extend the dominance transformation to work with the second-order B-format signal set.

Suppose that the transformation can be extended to accommodate the second-order component signals. The transformed signals must be linear combinations of the existing signals; hence, there must exist coefficients W', X', Y', U' and V', presumably functions of l, such that

$$A_2\cos(2q_2) = W'A_1 + X'A_1\cos(q_1) + Y'A_1\sin(q_1) + U'A_1\cos(2q_1) + V'A_1\sin(2q_1) \quad (6.77)$$

Note that it is sufficient to consider only the pantophonic case, since the periphonic case essentially reduces to this for $f = 0$. Complications due to the non-zero response of ***R*** for directions in the horizontal plane are avoided by using a notional signal which encodes only amplitude; whether this notional signal is in actuality proportional to ***W*** or to a combination of ***W*** and ***R*** is unimportant. The $1/\sqrt{2}$ scaling of ***W*** is also neglected for convenience.

We know that

$$A_2 = \frac{1}{2}\left[l\left(1+\cos(q_1)\right) + l^{-1}\left(1-\cos(q_1)\right)\right]A_1 \quad (6.78a)$$

$$\cos(q_2) = \frac{l^2 - 1 + \left(l^2+1\right)\cos(q_1)}{l^2 + 1 + \left(l^2-1\right)\cos(q_1)} \quad (6.78b)$$

$$\sin(q_2) = \frac{2l\sin(q_1)}{l^2 + 1 + \left(l^2-1\right)\cos(q_1)} \quad (6.78c)$$

and also that $A_2\cos(2q_2)$ can be found by using the identity

$$\cos(2q) = \cos^2(q) - \sin^2(q) \quad (6.79)$$

By taking various values of q, it is possible to generate a set of simultaneous equations which can then be solved for W', X', etc.

i) Let $q_1 = 0$. Then

$$A_2 = lA_1 \quad (6.80a)$$

$$q_2 = 0 \tag{6.80b}$$

and

$$\begin{aligned} A_2 &= W' A_1 + X' A_1 + U' A_1 \\ l A_1 &= W' A_1 + X' A_1 + U' A_1 \\ l &= W' + X' + U' \end{aligned} \tag{6.81}$$

ii) Let $q_1 = 180°$. Then

$$A_2 = l^{-1} A_1 \tag{6.82a}$$

$$q_2 = 180° \tag{6.82b}$$

and

$$\begin{aligned} A_2 &= W' A_1 - X' A_1 + U' A_1 \\ l^{-1} A_1 &= W' A_1 - X' A_1 + U' A_1 \\ l^{-1} &= W' - X' + U' \end{aligned} \tag{6.83}$$

iii) Let $q_1 = 90°$. Then

$$A_2 = \frac{1}{2}\left[l + l^{-1}\right] A_1 \tag{6.84a}$$

$$\cos(q_2) = \frac{l^2 - 1}{l^2 + 1} \tag{6.84b}$$

$$\sin(q_2) = \frac{2l}{l^2 + 1} \tag{6.84c}$$

and

$$
\begin{aligned}
A_2 \cos(2q_2) &= \frac{1}{2}\left[l + l^{-1}\right]A_1\left(\left[\frac{l^2-1}{l^2+1}\right]^2 - \frac{4l^2}{\left[l^2+1\right]^2}\right) \\
&= \frac{A_1}{2l}\left[l^2+1\right]\frac{l^4-6l^2+1}{\left[l^2+1\right]^2} \\
&= \frac{A_1}{2l}\frac{l^4-6l^2+1}{l^2+1}
\end{aligned}
\tag{6.85}
$$

so

$$
\begin{aligned}
\frac{A_1}{2l}\frac{l^4-6l^2+1}{l^2+1} &= W'A_1 + Y'A_1 - U'A_1 \\
\frac{1}{2l}\frac{l^4-6l^2+1}{l^2+1} &= W' + Y' - U'
\end{aligned}
\tag{6.86}
$$

iv) Let $q_1 = -90°$. Then

$$A_2 = \frac{1}{2}\left[l + l^{-1}\right]A_1 \tag{6.87a}$$

$$\cos(q_2) = \frac{l^2-1}{l^2+1} \tag{6.87b}$$

$$\sin(q_2) = \frac{-2l}{l^2+1} \tag{6.87c}$$

and

$$A_2 \cos(2q_2) = \frac{A_1}{2l}\frac{l^4-6l^2+1}{l^2+1} \tag{6.88}$$

so

$$
\begin{aligned}
\frac{A_1}{2l}\frac{l^4-6l^2+1}{l^2+1} &= W'A_1 - Y'A_1 - U'A_1 \\
\frac{1}{2l}\frac{l^4-6l^2+1}{l^2+1} &= W' - Y' - U'
\end{aligned}
\tag{6.89}
$$

v) Let $q_1 = 45°$. Then

$$\begin{aligned} A_2 &= \frac{1}{2}\left[l\left(1+\frac{1}{\sqrt{2}}\right) + l^{-1}\left(1-\frac{1}{\sqrt{2}}\right)\right] A_1 \\ &= \frac{1}{2\sqrt{2}}\left[l\left(\sqrt{2}+1\right) + l^{-1}\left(\sqrt{2}-1\right)\right] A_1 \\ &= \frac{\sqrt{2}}{4}\left[\sqrt{2}\left(l + l^{-1}\right) + \left(l - l^{-1}\right)\right] A_1 \end{aligned} \tag{6.90a}$$

$$\begin{aligned} \cos(q_2) &= \frac{l^2 - 1 + \left(l^2+1\right)/\sqrt{2}}{l^2 + 1 + \left(l^2-1\right)/\sqrt{2}} \\ &= \frac{\sqrt{2}\left(l^2-1\right) + \left(l^2+1\right)}{\sqrt{2}\left(l^2+1\right) + \left(l^2-1\right)} \\ &= \frac{\sqrt{2}\left(l - l^{-1}\right) + \left(l + l^{-1}\right)}{\sqrt{2}\left(l + l^{-1}\right) + \left(l - l^{-1}\right)} \end{aligned} \tag{6.90b}$$

$$\begin{aligned} \sin(q_2) &= \frac{2l/\sqrt{2}}{l^2 + 1 + \left(l^2-1\right)/\sqrt{2}} \\ &= \frac{2l}{\sqrt{2}\left(l^2+1\right) + \left(l^2-1\right)} \\ &= \frac{2}{\sqrt{2}\left(l + l^{-1}\right) + \left(l - l^{-1}\right)} \end{aligned} \tag{6.90c}$$

and

$$\begin{aligned} A_2 \cos(2q_2) &= \frac{\sqrt{2}}{4}\left[\sqrt{2}\left(l + l^{-1}\right) + \left(l - l^{-1}\right)\right] A_1 \\ &\qquad \times \left(\left[\frac{\sqrt{2}\left(l - l^{-1}\right) + \left(l + l^{-1}\right)}{\sqrt{2}\left(l + l^{-1}\right) + \left(l - l^{-1}\right)}\right]^2 - \frac{4}{\left[\sqrt{2}\left(l + l^{-1}\right) + \left(l - l^{-1}\right)\right]^2} \right) \\ &= \frac{\sqrt{2}A_1}{4}\left[\sqrt{2}\left(l + l^{-1}\right) + \left(l - l^{-1}\right)\right] \\ &\qquad \times \frac{2\left(l - l^{-1}\right)^2 + 2\sqrt{2}\left(l - l^{-1}\right)\left(l + l^{-1}\right) + \left(l + l^{-1}\right)^2 - 4}{\left[\sqrt{2}\left(l + l^{-1}\right) + \left(l - l^{-1}\right)\right]^2} \\ &= \frac{\sqrt{2}A_1}{4}\,\frac{2\left(l - l^{-1}\right)^2 + 2\sqrt{2}\left(l^2 - l^{-2}\right) + \left(l + l^{-1}\right)^2 - 4}{\sqrt{2}\left(l + l^{-1}\right) + \left(l - l^{-1}\right)} \end{aligned} \tag{6.91}$$

so

$$\frac{\sqrt{2}A_1}{4}\frac{2\left(l-l^{-1}\right)^2+2\sqrt{2}\left(l^2-l^{-2}\right)+\left(l+l^{-1}\right)^2-4}{\sqrt{2}\left(l+l^{-1}\right)+\left(l-l^{-1}\right)}=W'A_1+\frac{X'}{\sqrt{2}}A_1+\frac{Y'}{\sqrt{2}}A_1+V'A_1$$

$$\frac{\sqrt{2}}{4}\frac{2\left(l-l^{-1}\right)^2+2\sqrt{2}\left(l^2-l^{-2}\right)+\left(l+l^{-1}\right)^2-4}{\sqrt{2}\left(l+l^{-1}\right)+\left(l-l^{-1}\right)}=W'+\frac{X'}{\sqrt{2}}+\frac{Y'}{\sqrt{2}}+V' \qquad (6.92)$$

$$\frac{1}{2}\frac{2\left(l-l^{-1}\right)^2+2\sqrt{2}\left(l^2-l^{-2}\right)+\left(l+l^{-1}\right)^2-4}{\sqrt{2}\left(l+l^{-1}\right)+\left(l-l^{-1}\right)}=\sqrt{2}W'+X'+Y'+\sqrt{2}V'$$

Thus five simultaneous equations have been obtained, sufficient to determine the five coefficients. By appropriate manipulations we obtain

$$X'=\frac{1}{2}\left(l-l^{-1}\right) \qquad (6.93a)$$

$$Y'=0 \qquad (6.93b)$$

$$W'=\frac{l^2-2+l^{-2}}{2\left(l+l^{-1}\right)} \qquad (6.93c)$$

$$U'=\frac{2}{l+l^{-1}} \qquad (6.93d)$$

and

$$V'=\frac{2\left(l-l^{-1}\right)}{\left(\sqrt{2}+1\right)l^2+2\sqrt{2}+\left(\sqrt{2}-1\right)l^{-2}} \qquad (6.94)$$

However, if instead equation (6.92) is obtained by setting $q_1=-45°$, then we have instead

$$\frac{1}{2}\frac{2\left(l-l^{-1}\right)^2+2\sqrt{2}\left(l^2-l^{-2}\right)+\left(l+l^{-1}\right)^2-4}{\sqrt{2}\left(l+l^{-1}\right)+\left(l-l^{-1}\right)}=\sqrt{2}W'+X'-Y'-\sqrt{2}V' \qquad (6.95)$$

The sign reversal on Y' is of no consequence, since we still obtain a value of zero for Y' as before. However, the sign reversal on V' means that the solution is now

$$V'=\frac{-2\left(l-l^{-1}\right)}{\left(\sqrt{2}+1\right)l^2+2\sqrt{2}+\left(\sqrt{2}-1\right)l^{-2}} \qquad (6.96)$$

We have thus obtained a contradiction, since equations (6.94) and (6.96) cannot simultaneously be satisfied. Hence, it is not possible to find coefficients W', X', etc., independent of q, such that equation (6.77) is satisfied, and so it is not possible to extend the dominance transformation to the second-order case.